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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

In Union Is Strength

Many of the more active makers of the curricula of elementary and secondary schools have no high esteem for the values of mathematics and little conviction that it should be considered a universal and permanent element of the educational program. These individuals have taken a very active interest in seeking the answer to a question which should be of vital concern to every thinking adult in the United States, namely, "What are the characteristics and what is the content of that educational program which can do most good for each individual child in the United States?" Conscientious effort to answer this question very naturally throws into critical relief every instructional medium in use or which might be used to attain desirable educational objectives. Searching evaluation by such educational workers of the *raison d'être* of the mathematics curriculum in the elementary and secondary school, along with a more or less apathetic satisfaction in its *status quo* on the part of mathematicians and teachers of mathematics, has resulted in a tendency to consider mathematics of relatively minor importance in our system of general education. Those of us who are interested in mathematics and who are convinced of its educational potentialities need to bestir ourselves to actively challenge any trend to jeopardize its significance as a functional element in a program of general education.

The American Mathematical Society, the Mathematical Association of America and the National Council of Teachers of Mathematics are three organizations whose members have the interests of mathematics at heart. These groups should unite in earnest effort to acquaint the public with the nature and significance of mathematics as an asset to social progress, and to bring about a more effective understanding with educational workers for the better promotion of mathematics in general education. This is a job which should concern the research worker as well as the teacher. The next joint meeting of these three organizations will be at *Louisiana State University*, December 30, to January 3. May it be made an historic occasion through the deliberate union of purpose to maintain for mathematics that high public esteem and that professional respect and recognition it so richly deserves.

F. L. Wren, Second Vice-President, N. C. T. M.
George Peabody College for Teachers.

Polygons as Fundamental Elements in the Geometry of Plane Cubic Curves*

By J. M. FELD
New York City

1. *Introduction.* Let the points of a non-singular plane cubic C_3 be represented parametrically in terms of elliptic functions having 2ω and $2\omega'$ for primitive periods, so that the parameter 0 corresponds to one of its points of inflection. If a, b, c are the parameters of the vertices of a given inscribed triangle and k the parameter of a variable point, $a+k$, $b+k$, $c+k$ define a continuous manifold of ∞^1 triangles. It was shown by H. Oppenheimer† that these triangles satisfy theorems known to be satisfied by the points of C_3. In Oppenheimer's geometry two triangles associated with $k=k_1$ and $k=k_2$ determine uniquely a third belonging to the same system and associated with $k\equiv-(a+b+c+k_1+k_2)$, (mod 2ω, $2\omega'$); the third triangle here is the intersection of C_3 with the "secant" (a set of six lines) determined by the other two. The theorems on triangles (also called tripoints by Oppenheimer) of such a manifold and the "secants" determined by them are then shown by him to be valid also for polygons of a sort that is at once inscribed and circumscribed to C_3 and belong to a system given by $2^{p-1}(-1)^{p-1}a+h$, $p=0,1,2,\cdots,n-1$, where h is variable and a is fixed.

In this paper a new analogy between points on C_3 and polygons inscribed to C_3 is investigated: the polygons in question, unlike those used by Oppenheimer, being simple n-gons such that any two of them are cyclic-perspective. These serve as fundamental elements in place of points; sets of n^2 lines determined by pairs of these n-gons replace ordinary secants; and sets of curves C_p replace single curves intersecting C_3. In addition it is shown that any configuration bearing the symbol p_π, x_3, inscriptible to C_3 can generate an infinitude of inscriptible configurations having the symbol $(np)_{n\pi}, (n^2x)_3$.

2. *Cyclic-perspective n-gons.* Two simple n-gons in the projective plane, A_i, B_i, $i=0,1,2,\cdots,n-1$, that is, such for which a cyclic order has been posited, can have their vertices made to correspond

*Presented to the Society, April 7, 1939.

†H. Oppenheimer, *Uber Dreiecks-und Vieleckssysteme als Träger der Kurve dritter Ordnung*, Monatschefte für Mathematik und Physik, 20(1909), p. 141.

cyclically in $2n$ ways. These fall into two sets of n ways each; and in either set the correspondences occur by allowing the vertices of one n-gon to undergo cyclic permutations while those of the other are left alone. If the two n-gons are perspective in n ways by associating their vertices by means of such a set of cyclic permutations, E. Steinitz calls them *cyclic-perspective*.* The centers of perspectivity form a third n-gon C_i. Let A_k, B_l, C_m be collinear when $k+x+m\equiv 0$, mod n; then the $3n$ points of A, B and C and the n^2 lines $A_k B_l C_m$, $k+x+m\equiv 0$, mod n, form a regular configuration $(3n)_n$, $n_3{}^2$, the diagonals of which are sides of the complete n-points determined by the vertices of A, B and C. Steinitz showed (*loc. cit.*) that such configurations exist for all values of n; that their points lie on a C_3; and that conversely, on every real non-singular C_3 lie ∞^2 real configurations of this kind.† We shall call these configurations Steinitz configurations, and, with Steinitz, represent them by the symbol $[n]$.

Let A_k, B_l, C_m indiscriminately represent points on a real non-singular C_3 and the corresponding values of the elliptic parameter in terms of which C_3 is expressed. Let

$$\Delta=2p\omega/n+2q\omega'/n$$

where p and q are a pair of integers less than n and which do not have the same common divisior with n, so that $k\Delta\not\equiv 0$ when $k=1,2,\cdots,n-1$.

Since the number of ways of choosing such a pair of values for p and q is

$$N=n^2\left(1-\frac{1}{p_1{}^2}\right)\left(1-\frac{1}{p_2{}^2}\right)\cdots$$

where the p_i are the prime factors of n, N is the number of possible incongruent values of Δ for a given value of n.

For a given Δ we can associate with any point A_0 on C_3 a set of $n-1$ points $A_k\equiv A_0+k\Delta$ (mod 2ω, $2\omega'$), $k=1,2,\cdots,n-1$, constituting an n-gon A, which we shall describe as *derived* from A_0 by Δ. Let the secant $A_0 B_0$ intersect C_3 in the point C_0, and let $C_k\equiv C_0+k\Delta$ be the vertices of the n-gon derived from C_0. We assume first that C_0 is neither a vertex of A nor of B. Then, when $k+x+m\equiv 0$ (mod n), $A_k+B_l+C_m\equiv 0$ (mod 2ω, $2\omega'$). Since $A_{k+i}+B_{l-i}+C_m\equiv 0$ for $i=0,1,2,\cdots,n-1$,‡ A and B are perspective, the center of perspectivity being C_m. Evidently each pair of n-gons selected from A, B, C are

*E. Steinitz, *Uber Konfigurationen*, Archiv für Mathematik und Physik, (3) 16 (1910), p. 289.

†Such configurations exist also on nodal C_3's but not on cuspidal ones. Steinitz, *loc. cit.*

‡$A_k\equiv A_m$ (mod 2ω, $2\omega'$) when $k\equiv m$ (mod n).

cyclic-perspective, the centers of perspectivity being at the vertices of the third; therefore, the $3n$ vertices of A, B, C together with the n^2 lines $A_kB_lC_m$ constitute a Steinitz configuration $[n]$, which we shall describe as *derived* from the secant $A_0B_0C_0$. The same $[n]$ is derivable from each of its n^2 lines. Thus we can regard any inscribed $[n]$ as the analogue of a secant intersecting a C_3 in three distinct points; the cyclic-perspective n-gons and the n^2 lines joining them being analogues respectively of the ordinary points of intersection and their join. We shall call such a set of n^2 lines a *multiline*; the sets of cyclically ordered points of such n-gons *multipoints*; and three multipoints lying on the same multiline will be said to be *collinear*. Hence, *two distinct multipoints uniquely determine a multiline which intersects C_3 in a third multipoint; the latter, when distinct from the other two, being collinear with them.*

Let us now assume that C_0 coincides with a vertex of B, in which case the multipoint C derived from C_0 is a cyclic permutation of B, and therefore B may be regarded as cyclic-perspective to itself with the centers of perspectivity at the vertices of A. Under these conditions we shall call the multiline joining A and B the *tangent multiline* to C_3 at B, and, in analogy with ordinary usage, call A the *tangential multipoint* of B.

If A_0 is a point of inflection its derived multipoint A will be called a *multipoint of inflection*, since, under these circumstances, A is cyclic-perspective to itself and also its own tangential multipoint. The tangent multiline to C_3 at a multipoint of inflection is an inflectional tangent multiline, the component lines of which meet C_3 only in points of the inflectional multipoint. In general there are nine distinct multipoints of inflection, but for special values of n and Δ some may coincide. For instance, when $n=3$ and $\Delta=2\omega/3$ the inflectional multipoints derived from 0, $2\omega/3$ and $4\omega/3$ coincide. When the nine inflectional multipoints are distinct they lie by threes on twelve multilines, constituting four triangles of multilines, and may be described as a configuration $9_4,12_3$ of multipoints and multilines—the analogue of the Hessian configuration of nine inflectional points and twelve inflectional lines. From the nine harmonic polars nine $[n]$ can be derived; these are related to each other in a manner analogous to that of the ordinary harmonic polars.

Let the multiline x_1 meet C_3 in the three multipoints A_{i1}, A_{i2}, A_{i3} $(i=0,1,2,\cdots,n-1)$, which may be regarded as derived from the collinear points A_{01}, A_{02}, A_{03} respectively. Similarly, let the multiline x_2 meet C_3 in the multipoints B_{i1}, B_{i2}, B_{i3} derived from the collinear points B_{01}, B_{02}, B_{03}. The pairs of multipoints A_{i1}, B_{i1}; A_{i2}, B_{i2}; A_{i3}, B_{i3}

determine three multilines g_1, g_2, g_3 respectively, which intersect the cubic again in the multipoints C_{i1}, C_{i2}, C_{i3} respectively. Evidently lines $A_{01}B_{01}$, $A_{02}B_{02}$, $A_{03}B_{03}$ intersect C_3 in points belonging respectively to C_{i1}, C_{i2}, C_{i3}; representing these points by C_{01}, C_{02}, C_{03}, we observe that $C_{01}+C_{02}+C_{03}\equiv 0$ (mod 2ω, $2'\omega$) and that consequently the three multipoints $C_{ij}(j=1,2,3,)$ derived from these points are collinear on a multiline x_3. When the nine multipoints are distinct they lie in sets of three on six multilines constituting a configuration of multipoints and multilines to which we assign the symbol $9_2,6_3$. The totality of points and lines comprising the elements of this configuration form also a configuration of ordinary points and lines bearing the symbol $(9n)_{2n}$, $(6n^2)_3$. If x_3 and x_1 coincide x_2 may be termed the *satellite multiline* of x_1, in analogy with satellite of an ordinary line.

The three Hessian correspondences on C_3 determined by pairing every point u on the cubic with either $u+\omega$, $u+\omega'$, or $u+\omega+\omega'$ have analogues based on correspondences of multipoints: with each multipoint derived from u we can associate one derived from either $u+\omega$, $u+\omega'$, or $u+\omega+\omega'$.

3. *Multicurves.* The foregoing will serve to illustrate the analogy of the geometry of multipoints and multilines to that of ordinary points and lines on a cubic. The analogy can be extended to *multicurves* which intersect a C_3 in a complete set of multipoints. Let u_i, $i=1, 2,\cdots, 3p$, be a complete set of points on C_3 determined by a C_p; the set of $3p$ multipoints derived from these points is defined as a *complete set of multipoints.* The character of the theorems involving complete sets of multipoints can be ascertained by the consideration of a special case; we shall therefore limit ourselves to an examination of the case $p=2$, which implies that $\sum_{i=1}^{i=6} u_i\equiv 0$ (mod 2ω, $2\omega'$). The multipoint derived from u_i is given by $u_i+k\Delta$, $k=0,1,2,\cdots, n-1$, and will be represented by U_i.

Let us select one point from each of five of the multipoints, say U_1, U_2, U_3, U_4, U_5; the sum of the parameters of these points is $u_1+u_2+u_3+u_4+u_5+s\Delta$, $0\leqq s\leqq n-1$. There exists a unique point in U_6, $u_6+t\Delta$ where $t+s\equiv 0$, mod n, residual to the other five. Consequently, ***a conic which contains a point of each one of five in a partial set of multipoints contains also a point of the residual sixth; the*** $6n$ ***points of the complete set of multipoints therefore lie by sixes on*** n^5 ***conics.*** This manifold of conics we shall call a ***multiconic*** and represent it by C_2^n. Through each point of U_i pass n^4 conics of the multiconic. The elements of the multiconic and the six multipoints, when they are distinct, constitute a curvilinear configuration of $6n$ points and n^5 conics to

which the symbol $(6n)^4_n$, n^5_6 can be appropriately assigned. If two of the multipoints U_1 and U_2 coincide the multiconic C_2^n will be said to be tangent to C_3 at U_1; and in this event each constituent conic of C_2^n intersects C_3 in two points, distinct or coincident, of U_1. If U_2 and U_3 coincide with U_1 then C_2^n will be said to have contact of the second order with C_3 in which case each element of C_2^n intersects C_3 in three points belonging to U_1. A multiconic is determined by five multipoints; if three of these are collinear the multiconic degenerates into two multilines.

4. *The generation of configuration inscriptible in a cubic.* Let us consider a configuration K of p points and x lines, bearing the symbol p_λ, x_3, and inscribed to C_3; let us assume, as we may, that n and Δ are such that the p multipoints derived from the points of K are distinct. Since the multipoints derived from three points of K lying on a line of K are collinear, they can be joined by a multiline to form a Steinitz $[n]$ derived from this line. In this manner all the points and lines of K give rise to and can be replaced by distinct multipoints and multilines constituting a larger configuration of points and lines bearingt he symbol $(np)_{\lambda n}$, $(xn^2)_3$. Since an infinitude of values can be assigned to n an infinite number of such configurations can be generated from K. When K is a Steinitz $[p]$, however, the new figurations generated from it are merely larger Steinitz configurations $[np]$; new types of configurations appear, therefore, when the basic one is not a Steinitz.

A Note on Observed Geometric Series

By A. B. Soble
Philadelphia, Pa.

More logical, and more compensatory, than the moment method of calculating the ratio in an observed geometric series, is the following summation method, which seems to be original.

Let the observations be $y_1, y_2, \cdots, y_n$, and let the ratio be r. Let also

$$t_1 = \sum_1^{n-1} y_i \quad , \quad t_2 = \sum_2^{n} y_i$$

$$t_3 = \sum_1^{n-1} 1/y_i \quad , \quad t_4 = \sum_2^{n} 1/y_i.$$

Now $$t_2 \doteq r t_1$$

and $$t_3 \doteq r t_4.$$

Hence by least squares

$$r = \frac{t_1 t_2 + t_3 t_4}{t_1^2 + t_4^2} \cdot$$

We illustrate with the U. S. annual production of cigarettes from 1901 to 1913, which Day shows is approximately a geometric series.*

Year	y_i†	$1/y_i$		
1901	2.72	.3676		
1902	2.96	.3378		
1903	3.36	.2976	$t_1 = 70.74$	$t_3 = 2.5872$
1904	3.43	.2915	$t_2 = 83.58$	$t_4 = 2.2839$
1905	3.67	.2725		
1906	4.50	.2222		
1907	5.26	.1901	$t_1 t_2 = 5912.45$	$t_1^2 = 5004.15$
1908	5.74	.1742	$t_3 t_4 = 5.91$	$t_4^2 = 5.22$
1909	6.82	.1466		
1910	8.64	.1157	5918.36	5009.37
1911	10.47	.0955		
1912	13.17	.0759		
1913	15.56	.0643	$r = 1.181$	

Hence the trend of cigarette production for the years 1901-1913 was an annual increase of approximately 18%.

*Day, Edmund E., *Statistical Analysis*, 1930, p. 265.
†Unit: one billion cigarettes; *ibid.*, p. 275.

Root Isolation Through Curve Analysis

By E. C. Kennedy
Texas College of Arts and Industries

The real roots of $f(x)=0$, a real algebraic equation of degree n, can often be partially isolated by means of Budan's Theorem, Descartes' Rule of Signs, or by examining the value of $f(x)$ for various values of x. In the case of the cubic the real roots can be readily isolated by examining the bend points of the function. All of these schemes have the disadvantage that usually a root can be isolated only in a wide interval, e. g. it may be determined that one lies between 0 and $+\infty$, or else we are left in doubt as to the exact number of roots in an interval. It is desirable to have a practical method whereby each root may be isolated quickly in a relatively small interval. With this in view the following discussion is presented.

Consider the real cubic

$$X^3+AX^2+BX+C=0. \qquad \text{Write} \tag{1}$$

$$Y=-\frac{AX^2+C}{X^2+B} \qquad C\neq BA. \tag{2}$$

Noting that (2) becomes (1) when Y is replaced by X, we observe that the real roots of (1) are given by the intersection of the graph of (2) and the line $Y=X$. If $C=BA$, the cubic is readily factored and the solution is immediate. If $B<0$, then the curve will consist of three distinct branches. One is an arch with vertex at

$$P\left(0,-\frac{C}{B}\right),$$

sides asymptotic to $X=\pm\sqrt{-B}$, and symmetric with respect to the Y-axis. The rest of the curve consists of two branches which are asymptotic to $X=\pm\sqrt{-B}$ and to $Y=-A$. The curve intersects the X-axis at

$$X=\pm\sqrt{-\frac{C}{A}}$$

(if the radicand is positive), and these points may be either on the arch or on the other branches, but not on both, since for any fixed

value of Y there can be at most two real values of X. If $A=0$, the curve never cuts the $X-$axis but is asymptotic to it. If $C=0$, we solve the cubic by factoring.

The scheme is illustrated graphically for the case $C>0$; $A,B<0$ in the adjoining figure.

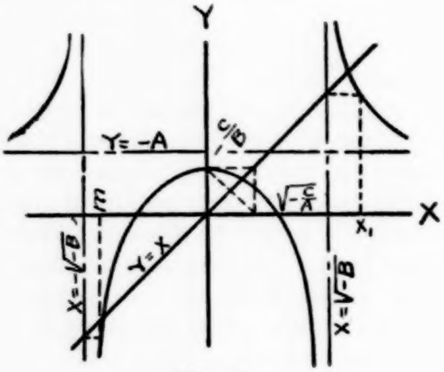


Fig. 1.

Note that if

$$-A > -\frac{C}{B}$$

the arch opens downward and cuts the $X-$axis.

Labeling the real roots R_1, R_2, R_3 in order of increasing magnitude, we have from the figure

$$-\sqrt{-B} < R_1 < -\sqrt{\frac{-C}{A}}$$

$$-\frac{C}{2B} < R_2 < -\frac{C}{B}, \qquad \text{if } -\frac{C}{B} \geqslant \sqrt{-\frac{-C}{A}}$$

$$\frac{1}{2}\sqrt{\frac{-C}{A}} < R_2 < \sqrt{\frac{-C}{A}}, \qquad \text{if } \sqrt{\frac{-C}{A}} < \frac{-C}{B}$$

$R_3 >$ either $-A$ or $\sqrt{-B}$.

By way of illustration let us consider

Example I. Isolate the roots of

$$X^3 - X^2 - 10X + 4 = 0. \tag{3}$$

From Fig. I and the above results we write

$$-\sqrt{10} < R_1 < -2$$

$$\frac{2}{10} < R_2 < \frac{4}{10}$$

$$\sqrt{10} < R_3.$$

To get an upper bound, X_1, for R_3 we set $Y=\sqrt{10}$ in (2) obtaining

$$X^2 = -\frac{YB+C}{Y+A} = -\left[\frac{-10\sqrt{10}+4}{\sqrt{10}-1}\right] = \frac{27.62}{2.162} = 12.78 \tag{4}$$

from which we get $X_1 = 3.57$.

Then using this value for Y in (4) we get

$$X^2 = -\frac{-35.7+4}{3.57-1} = 12.33 \text{ or } X_2 = 3.51,$$

a better lower bound for R_3. Repeated application of (4) gives increasingly better upper and lower bounds for R_3. Similarly, we can get arbitrarily accurate bounds for the other two roots by a repeated application of (2) and (4). This "closing down" process is sometimes very rapid.

If $B>0$, there are no vertical asymptotes. In this case the curve (2) is a continuous one and is completely described by the adjoining graph ($A<0$; $B,C>0$). If

$$\sqrt{\frac{-C}{A}} > -A,$$

the cubic obviously has only one real root and it is negative. The

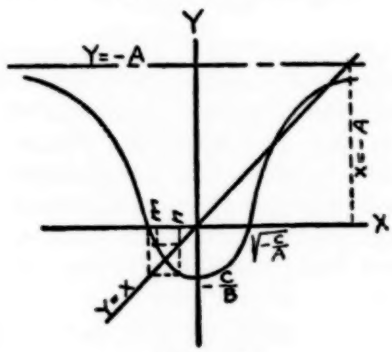


Fig. 2.

converse is not necessarily true. In doubtful cases one should examine the discriminant, Δ. If $\Delta \geqslant 0$, there are three real roots and we have

$$\frac{-C}{B} \text{ or } -\sqrt{\frac{-C}{A}} < R_1 < 0$$

$$\sqrt{\frac{-C}{A}} < R_2 \geqslant R_3 < -A.$$

Example II. Isolate the roots of

$$X^3 - 4X^2 + X + 5 = 0. \tag{5}$$

Here we have $$-1.12 < -\sqrt{\frac{5}{4}} < R_1 < 0.$$

From the figure we cannot say whether or not the two remaining roots are real, but if they are

$$1.11 = \sqrt{\frac{5}{4}} < R_2 \geqslant R_3 < 4.$$

Here again our bounds may be sharpened arbitrarily.

The above method of isolating the roots of a cubic can usually be carried out very quickly. All we need to know is where the curve (2) cuts the axes, and its asymptotes. This gives us different information to sketch in roughly our curve and read off the results. The case where $B = 0$ is easily handled in much the same manner.

Isolate the roots of the real quartic

$$X^4 + AX^3 + BX^2 + CX + D = 0. \tag{6}$$ Write

$$Y = -\frac{X^2 + BX^2 + D}{AX^4 + C} \, . \tag{7}$$

Since the numerator and denominator are in the form of a quadratic, we can readily find the asymptotes and where the curve cuts the X-axis. It is then a case of sketching in the auxiliary curve (along with the curve $Y = X$) and reading off our results. Here again if the denominator is a factor of the numerator, the quartic may be solved by factoring. Special cases arise when certain coefficients are zero, but these cases present no difficulty and lack of space forbids a full discussion of them.

(NOTE: At times it might be advantageous to write (7) in another form.)

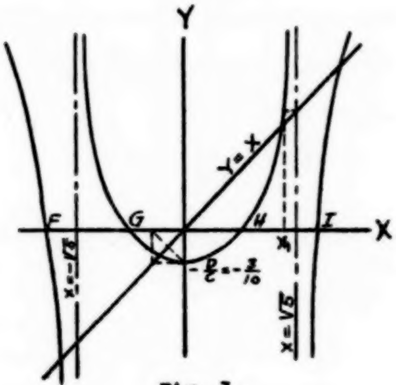


Fig. 3.

As an illustrative example consider the figure above which is associated with

$$X^4-2X^3-12X^2+10X+3=0.$$

Here
$$F=-3.43<R_1<-\sqrt{5}<=2.28$$
$$G=-\ .30<R_2<-\ .15$$
$$H=\ \ .505<R_3<\sqrt{5}<2.24$$
$$I=\ \ 3.43<R_4.$$

As in the case of the cubic, these bounds may be arbitrarily sharpened. The method of procedure is obvious.

To isolate the roots of the real quintic

$$X^5+AX^4+BX^3+CX^2+DX+E=0 \tag{8}$$

$$Y=-\frac{AX^4+CX^2+E}{X^4+BX^2+D}\ . \tag{9}$$

and, as before, the abscissas of the points of intersection of (9) and $Y=X$ gives the roots of (8). As in the case of the cubic and the quartic many special cases arise, but they are easily disposed of.

With a little practice one can very quickly isolate the roots of a cubic, quartic, or quintic by the above scheme, but equations of higher degree can be readily handled only in more or less special cases. It often happens that multiple roots do not show up as such on the figure.

For example, our analysis shows that $16X^4-24X^2+16X-3=0$ has exactly two distinct real roots. It would be wrong to infer that the remaining roots are complex for one of the two real roots is a triple root.

Our method will usually prove useful on such equations as

$$X^m(X-A_1)(X-A_2)\cdots(X-A_k)-(X-B_1)(X-B_2)\cdots(X-B_e)=0.$$

For example, we can readily prove that

$$X^m(X-3)(X+4)-(X-1)(X+2)=0$$

has three roots: $-4<R_1<-2$; $0<R_2<1$; $3<R_3$, if m is odd, and four real roots: $R_1<-4$; $-2<R_2<0$; $0<R_3<1$; $3<R_4$ if m is even. The method of procedure is evident.

The trinomial equation

$$X^m-AX^n-B=0 \tag{10}$$

deserves a word or two. Writing $Y_1=X^n$, $Y_2=B/(X^{m-n}-A)$ and examining the graphs of these two equations we find that if $A, B>0$, then (10) has just one positive root, R, and it satisfies the inequalities

$$A<R^{m-n}<A+\frac{B}{A^{n/m-n}}\ . \tag{11}$$

These bounds may be sharpened to any desired degree. Again, lack of space prevents a complete discussion of the various cases that arise.

As an exercise in curve analysis perhaps the reader would like to work the following problems.

Problem 1. Isolate the positive root of $X^{30}-100X^{28}-500{,}000=0$ in an interval of measure .00000 00000 00000 00000 0003.

Problem 2. Show that $X^7-2X^2+3X^6-27=0$ has exactly one positive root, R, and that this root satisfies the equalities

$$2<R<\sqrt[3]{9}<2.08.$$

Probability DeLuxe. A contract bridge expert writing for a Cleveland paper was analysing the relative merits of (1) finessing against the queen, and (2) playing for the "break", in the case where the declarer holds the ace, king, jack of trumps. He arrived at the conclusion that the probability of plan (1) working was the same as the probability of plan (2) being successful. Then he made the startling observation, "However, in actual play one should adhere always to just one plan. For, in playing for the break on some occasions and taking the finesse on others, it is possible to guess wrong an abnormal number of times. On the other hand always using the same plan of play gives the law of probility a chance to operate".

probity

A Note on the Distribution of the Median

By EDWARD PAULSON
New York City

An expression for the distribution of the median of a sample containing an odd number of items from any population was given by Craig;* it was also given in a slightly different form by Hojo.† Although the actual distribution of the median is usually of extreme complexity, it will be found that it is a relatively easy matter to test the significance of the sample median and to establish fiducial limits for the mean. Some equations superficially resembling those to be given later occur in an interesting paper by Thompson‡ who, however, discusses the matter entirely from the viewpoint of inverse probability. Although this approach did not necessitate any assumption concerning an equal distribution of ignorance, still the results did not appear to be of practical value. The simple results given below, however, would appear to considerably increase the value of the median in sampling from populations such as the Cauchy distribution

$$\frac{dx}{\pi[1+(x-m)^2]}$$

for which the mean is not a consistent statistic, and in general in small samples where there is a priori reason for believing that the population sampled is non-normal.

Only the case of a symmetric parent population will now be considered.

Let $f(x)dx$ be the probability element of a population, so that

$$f(x)=f(-x) \qquad (-a \leqq x \leqq a)$$

and the mean and median are both zero.

The distribution of the median M of samples of $(2n+1)$ random observations is

$$\Theta(M)dM=\frac{(2n+1)!}{n!\ n!}\left[\int_{-a}^{M} f(x)dx\right]^n\left[\int_{M}^{a} f(x)dx\right]^n \times f(M)dM,$$

*Allen T. Craig, *American Journal of Mathematics* (1932), Vol. 54, p. 364.
†Tokishige Hojo, *Biometrika*, Vol. 23 (1931-1932) p. 317.
‡William R. Thompson, *Annals of Mathematical Statistics*, 1936, pp. 122-128.

Let $$y=\int_{-a}^{M} f(x)dx,$$

$$\frac{dy}{dM}=f(M). \qquad \left\langle \begin{matrix} M=a, & y=1 \\ M=-a, & y=0 \end{matrix} \right\rangle$$

Making use of the relation

$$\int_{M}^{a} f(x)dx=1-\int_{-a}^{M} f(x)dx,$$

the distribution of y is determined by

$$\Theta(M)dM=G(y)dy=\frac{(2n+1)!}{n!\ n!}\ y^n(1-y)^n dy. \quad (0\leqq y\leqq 1)$$

Let $$F(x)=\int_{0}^{x} f(t)dt.$$

Since $$f(x)=f(-x), \quad F(x)=-F(-x).$$

$$\int_{0}^{a} f(x)dx=\tfrac{1}{2}=F(a).$$

$$\therefore y=F(M)-F(-a)=F(M)+\tfrac{1}{2}.$$

Since the mean value of M is 0, the probability of getting in random sampling a value of the median which shall have a greater deviation from zero than the observed sample value M_0 is

$$P(|M|\geqq|M_0|)=2\int_{|M_0|}^{a} \Theta(M)dM=2\int_{F(|M_0|)+1/2}^{1} G(y)dy$$

$$=\frac{2(2n+1)!}{n!\ n!} \times \int_{F(|M_0|)+1/2}^{1} y^n(1-y)^n dy.$$

The expression under the integral sign is the incomplete beta

function, which has been tabulated* in the form

$$I_x(P, q) = \frac{B_x(p,q)}{B(p,q)} = \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q)} \int_0^x x^{p-1}(1-x)^{q-1}dx.$$

$$\therefore P(|M| \geqq |M_0|) = 2[1 - I_{F(|M_0|)+1/2}(n+1,\ n+1)].$$

To test the hypothesis that a given sample of $(2n+1)$ items with a median at M_0 is a random sample from a certain symmetrical population $f(x-m)dx$ with a mean at $m=m_0$, all other parameters being specified—it is only necessary after letting $y=x-m$ to compute

$$\alpha = \int_0^{|M_0-m_0|} f(y)dy$$

correct to several decimal places and then to evaluate by means of tables

$$P = 2 - 2I_{\alpha+1/2} \quad (n+1,\ n+1).$$

The fiducial limits for the mean found through the use of the median will not in general be of unique value, but they will often provide useful information. To find the fiducial limits for the mean m of a symmetrical population $f(x-m)dx$ on a level of probability P_1, let

$$y = x - m \quad \text{and} \quad F(y) = \int_0^y f(y)dy.$$

The fiducial limits for m are

$$M_0 \pm t$$

where t is found from the equation

$$z = F(t) + \tfrac{1}{2}$$

z being determined from the relation

$$I_z(n+1,\ n+1) = 1 - \frac{P_1}{2}$$

The preceding analysis, although restricted to symmetrical distributions, includes most of the cases likely to be of practical importance. In the case of skew distributions the problem of using the median to obtain information about the population mean is more

*_Tables of the Incomplete Beta Function_, edited by Karl Pearson, 1934.

complicated and perhaps somewhat artificial; moreover it is best restricted to distributions whose range is infinite at both ends, otherwise as in the distribution $(x-m)e^{-(x-m)}dx(m<x<\infty)$, there may be

very poor control over the type of error which results in accepting a hypothesis as true when it is in fact false. It is possible, however, when this restriction is satisfied to transform the problem into the one previously considered for symmetrical distributions by using a new variate

$$z=\int_{-\infty}^{x} f(x-m)dx-\tfrac{1}{2},$$

for z will now be symmetrically distributed with uniform density from $-\frac{1}{2}$ to $+\frac{1}{2}$.

LITERATURE RECEIVED BY THE EDITORIAL BOARD DURING PERIOD

JULY 1, 1939-APRIL 1, 1940

(Volume XIV)

(1) *Advanced Calculus.* By Ivan S. Sokolnikoff. McGraw-Hill Book Company, New York. x+446 pages. $4.00.

(2) *A Short History of Science.* By Sedgwick, Tyler, Bigelow. The Macmillan Company. New York. $3.75.

(3) *College Algebra.* By Paul R. Rider. The Macmillan Company, New York. viii+372 pages. 1940. $2.00.

(4) *College Algebra.* By Charles H. Sisam. Henry Holt and Company, New York, 1940. xii+395 pages.

(5) *Development of the Minkowski Geometry of Numbers.* By Harris Hancock. The Macmillan Company, New York. $12.00.

(6) *Elementary College Mathematics.* By Ernest Lloyd Mackie and Vinton Asbury Hoyle. Ginn and Company, Boston. 1940. $2.80.

(7) *Elementary Theory of Equations.* By William Vernon Lovitt. Prentice-Hall Inc. xi+237 pages. $2.50.

(8) *Mathematical Methods in Engineering.* By V. Karman and D. Biot. McGraw Hill Book Company, New York. 1940. $4.00.

(9) *Reviews and Examinations in Algebra.* By Oswald Tower and Winfield M. Sides. D. C. Heath and Company, New York. $1.20.

(10) *Solution of a Cubic Diophantine Equation.* By W. V. Parker and A. A. Aucoin. Reprint from the Tohoku Mathematical Journal, volume 45, part 2, March, 1939.

7 (11) *The First Course in College Mathematics.* By W. E. Anderson. Harper Bros., New York. $12.00.

(12) *Three Copernican Treatises.* Translated with introduction and notes. By Edward Rosen. Columbia University Press. New York. x+211 pages. $3.00.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from March issue.)

It seems that the discovery of the principles as well as his two demonstrations of Problems 21 and 23 referred to on page 327 of this Magazine were original with Professor Adrain.*

We shall give Professor Adrain's two demonstrations of this law in full.

ARTICLE XIV.

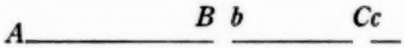
Researches concerning the probability of the errors which happen in making observations, etc.

By ROBERT ADRAIN

The question which

I propose to resolve is this: Suppose AB to be the true value of any quantity, of which the measure by observation or experiment is Ab, the error being Bb; what is the expression of the probability that the error Bb happens in measuring AB?

Let AB, BC, etc. be several successive distances of which the



values by measure are Ab, bc, etc., the whole error being Cc: now supposing the measures Ab, bc, to be given and also the whole error Cc, we assume as an evident principle that the most probable distances

*For a resumé of various demonstrations of the "Method of Least Squares", see Hendrick's *Analyst*, Vol. IV, No. 2, 1877.

AB, BC are proportional to the measures Ab, bc; and therefore the errors belonging to AB, BC are proportional to their lengths, or to their measured values, Ab, bc. If, therefore, we represent the values of AB, BC, or of their measures Ab, bc, by a, b, the whole error Cc by E, and the errors of the measures Ab, bc by x, y, we must, for the greatest probability, have the equation

$$\frac{x}{a} = \frac{y}{b} .$$

Let X and Y be similar functions of a, x, and b, y the probabilities that the errors x, y, happen in the distances a, b; and, by the fundamental principle of the doctrine of chance, the probability that both these errors happen together will be expressed by the product XY. If now we were to determine the values of x and y from the equations $x+y=E$, and $XY=$maximum, we ought evidently to arrive at the equation

$$\frac{x}{a} = \frac{y}{b} :$$

and since x and y are rational functions of the simplest order possible of a, b, and E, we ought to arrive at the equation

$$\frac{x}{a} = \frac{y}{b}$$

without the intervention of roots, in other words by simple equations; or, which amounts to the same thing in effect, if there be several forms of X and Y that will fulfill the required condition, we must choose the simplest possible, as having the greatest possible degree of probability.

Let X', Y', be the logarithms of X and Y, to any base or modulus e: and when $XY=$max. its logarithm $X'+Y'=$max. and therefore $dX+dY=0$, which fluxional equation we may express by

$$X''dx+Y''dy=0;$$

for as X' involves only the variable quantity x, its fluxion dX' will evidently involve only the fluxion of x; in like manner the fluxion of Y' may be expressed by $Y''dy$; but since $x+y=E$ we have also $dx+dy=0$, and $dx=-dy$, by which dividing the equation $X''dx=-Y''dy$, we obtain $X''=Y''$.

Now this equation ought to be equivalent to

$$\frac{x}{a} = \frac{y}{b} ;$$

and this circumstance is effected in the simplest manner possible, by assuming

$$X'' = \frac{mx}{a}, \text{ and } Y'' = \frac{my}{b};$$

m being any fixed number which the question may require.

Since therefore

$$X'' = \frac{mx}{a},$$

we have

$$X''dx = dX' = \frac{mxdx}{a},$$

and taking the fluent, we have

$$X' = a' + \frac{mx^2}{2a},$$

the constant a' being either absolute, or some function of the distance a.

We have discovered therefore, that the logarithm of the probability that the error x happens in the distance a is expressed by

$$a' + \frac{mx^2}{2a} = X',$$

and consequently the probability itself is

$$X = e^{X'} = e^{\left(a' + \frac{mx^2}{2a}\right)}.$$

Such is the formula by which the probabilities of different errors may be compared, when the values of the determinate quantities e, a', and m are properly adjusted. If this probability of the error x be denoted by u, the ordinate of a curve to the abscissa x, we shall have

$$u = e^{\left(a' + \frac{mx^2}{2a}\right)},$$

which is the general equation of the *curve of probability*.

When only the maximum of probability be required, we have no need of the values of e, a', and m; it is proper, however, to observe that m must be negative. This is easily shown. The probability

that the errors x, y, z, etc. happen in the distances a, b, c, etc. is

$$e^{\left(a'+\frac{mx^2}{2a}\right)}\times e^{\left(b'+\frac{mx^2}{2b}\right)}\times e^{\left(c'+\frac{mx^2}{2c}\right)} \text{ etc.}$$

which is equal to

$$e^{\left(a'+b'+c'+\,.\,.\,+\frac{mx^2}{2a}+\frac{my^2}{2b}+\frac{mz^2}{2c}+\,.\,.\right)}$$

and this quantity will evidently be a maximum or minimum as its index or logarithm

$$a'+b'+c'+\cdots+\frac{mx^2}{2a}+\frac{my^2}{2b}+\frac{mz^2}{2c}+\cdots$$

is a maximum or minimum, that is, when

$$\frac{m}{2}\left\{\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\cdots\right\}=\text{a maximum or minimum.}$$

Now when $x+y+z+\cdots\cdot=E$, we know that

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}+\cdots=\text{min. when}$$

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\cdots$$

and therefore $$-\left\{\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\cdots\right\}\text{is a maximum,}$$

when $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\cdots.$$

It is evident, therefore, that m must be negative, and as we may for the case of maxima use any value of it we please, we may put $m=-2$, and the probability of x in a is

$$u=e^{\left(a^1-\frac{x^2}{a}\right)}.$$

If we put $$\frac{m}{2a}=-1 \text{ and } a'=f^2,$$

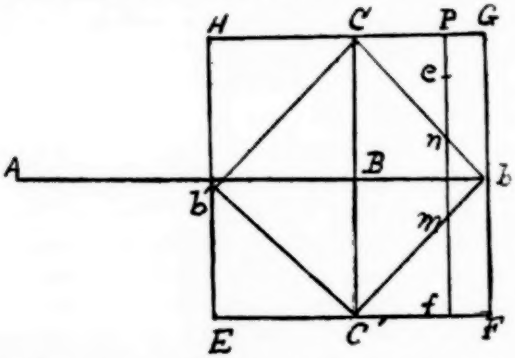
we have $u=e^{(f^2-x^2)}$ for the equation of the curve of probability; but if we suppose $f^2=0$, the ordinates u will still be proportional to their former values and we shall have $u=e^{-x^2}$, or

$$u=\frac{1}{\epsilon}x^2,$$

which is the simplest form of the equation expressing the nature of the curve of probability.

We shall now confirm what has been said, by a different method of investigation.

Suppose that the length and bearing of AB are to be measured; and that the little equal straight lines Bb, BC are the equal probable errors, the one $Bb=Bb'$ of the length of AB, the other $BC=BC'$ (perpendicular to the former) of the angle at A, when measured on a



circular arc to the radius AB: and let the question be to find such a curve passing through the four points b, C, b', C', which are equally distant from B, that, supposing the measurement to commence at A, the probability of terminating on any one of the four points b, C, b', C'.

Describe the squares $bCb'C'$, $EFGH$. I say the curve sought must pass within the greater square $EFGH$, but without the less square $bCb'C'$.

Let $mnep$ be drawn parallel to BC; and since the probabilities of the indefinitely little equal errors BC, mP, are ultimately in the ratio of equality; but the probability of the error Bm in the distance is less than the probability of the error 0 at B, (for it is self evident that the greater the error is, the less is its probability) therefore, by the laws of chance, the probability of terminating on P is less than that of terminating on C, and therefore the point P is without the curve sought.

By the same argument we prove that bG is without the curve.

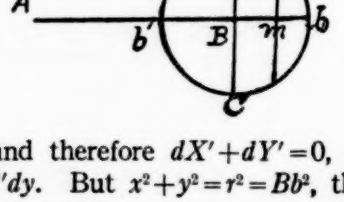
Again, since the sum of the two errors Bm, mn, in distance and bearing, is together equal to the error Bb, it follows that the probability of terminating on n is greater than that on b; for it certainly is more reasonable to suppose that each of two equal sources of error should produce a part of the whole error $Bm+mn=Bb$, than that the whole error Bb should be produced from one of these sources alone, without any assistance from the other.

The same thing may also be shown thus, the probability of mn is the same as if it were reckoned on BC from B, and therefore the probability of mn is greater than that of mb, because any particle of error in Bb or BC is always less probable as that particle is farther from the point B, of course the point n is within the curve; and therefore the curve must fall without the square $bCb'C'$. This curve therefore passes through the four points b, C, b', C' equally distant from B, and lies in the four triangles bGC, CHb', $b'EC'$, and $C'Fb$.

Further, the curve in question ought evidently to be continuous, and have its four portions similar which lie in the four triangles bGC, CHb', etc. Its arcs proceeding from b to C or from C to b ought to be similar to each other, and to each of these proceeding from C, b', and C'. It must have two and only two ordinates me, mf to the same abscissa Bm; and those ordinates must be equal, the one positive, the other negative. The value of the ordinate must be the same whether the error Bm be positive or negative, that is, in excess, or defect. The equation of the curve must therefore have two equal values of the ordinate $y=ne=nf$ to the same abscissa $=x$; and the abscissa x must have two equal values to the same value of the ordinate y. Lastly, the curve must be the simplest possible, having all the preceeding conditions, and must consequently be the circumference of a circle having its center in B.

Now let us investigate the probability of the error $Bm=x$, and of $mn=y$.

Let X and Y be two similar functions of x and y denoting those probabilities, X', Y' their logarithms, then $XY=$constant, or



$X'+Y'=$constant, and therefore $dX'+dY'=0$, or $X''dx+Y''dy=0$, whence $X''dx=-Y''dy$. But $x^2+y^2=r^2=Bb^2$, therefore $xdx=-ydy$,

by which dividing $X''dx = -Y''dy$, we have

$$\frac{X''}{x} = \frac{Y''}{y} \quad ;$$

and therefore, by a fundamental principle of similar functions, the similar functions

$$\frac{X''}{x} = \frac{Y''}{y}$$

must be each a constant quantity: put

$$\frac{X''}{x} = n,$$

and we have $X''dx = nxdx$, that is, $dX = nxdx$, and the fluent is

$$X' = C + \frac{nx^2}{2} \quad ;$$

in like manner, we find $\qquad Y' = C + \dfrac{ny^2}{2}$,

and therefore the probabilities themselves are

$$e^{C+\frac{nx^2}{2}} \text{ and } e^{C+\frac{ny^2}{2}},$$

in which n ought to be negative, for the probability of x grows less as x grows greater.

If now put the constant quantities C and n equal to a', and

$\dfrac{m}{2a}$, we have $u = e^{\left(a' + \frac{mx^2}{2a}\right)}$ as before.

As the application of this formula to maxima and minima does not require the value of a' or m we shall suppose $a' = 0$, and $m = -2$, in which case we have $u = e^{\frac{-x^2}{a}}$.

If there be only one quantity to which the errors relate, we may put $C = 0$, and

$$\frac{n}{2} = -1,$$

in which case $u=e^{-x^2}$, in which u is the probability of the error x, or the ordinate of the curve of probability to the abscissa x.

Suppose now that the equally probable errors Bb, Bc are in any proposed ratio of 1 to p; let Bb and Bc be expressed by x and X, and supposing that $e^{\frac{-x^2}{a}}$ is the probability of x, I say the probability of X will be $e^{\frac{-X^2}{p^2a}}$.

For, since $1:p=x:X$, therefore

$$x=\frac{X}{p},\quad x^2=\frac{X^2}{p^2},\text{ and } e^{-\frac{x^2}{a}} = \text{etc.} =e^{-\frac{x^2}{p^2a}}.$$

In this case, the curve of equal probability is an ellipse. We shall now show the use of this theory in the solution of the following problems:

PROBLEM I.

Supposing a, b, c, d, etc. *to be the observed measures* of any quantity x, the most probable value of x is required.

Solution: The several errors are $x-a$, $x-b$, $x-c$, $x-d$, etc. and the logarithms of their probabilities are, by what has just been shown, $-(x-a)^2$, $-(x-b)^2$, $-(x-c)^2$, $-(x-d)^2$, etc.; therefore,

$$(x-a)^2+(x-b)^2+(x-c)^2+(x-d)^2+\text{etc.}=\text{min.}$$

The fluxion of this divided by $2dx$ gives us

$$x-a+x-b+x-c+x-d+\text{etc.}=0:$$

let n be the number of terms and we have $nx=a+b+c+d+$etc. to n terms; therefore

$$x=\frac{a+b+c+d+\cdots}{n}=\frac{s}{n},$$

putting s for the sum of a, b, c, etc. Hence the following rule:

Divide the sum of all the observed values by their number, and the quotient will be the most probable value required.

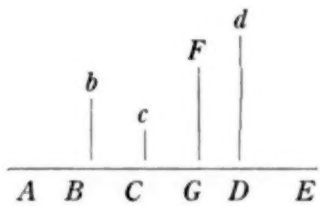
This rule coincides exactly with that commonly practiced by astronomers, navigators, etc.

It is worthy of notice, that according to the solution given above, the sum of all the errors in excess is precisely equal to the sum of all the errors in defect; in other words, the sum of all the errors is precisely equal to 0, each error being taken with its proper sign: this is evident from the equation $(x-a)+(x-b)+(x-c)+(x-d)+\text{etc.}=0$.

PROBLEM II.

Given the observed position of a point in space, to find the most probable position of the point.

Solution: On any fixed plane let fall perpendiculars from all the points of position, and also from the point sought, meeting the plane in b, c, d, etc. and in F, and on the straight line AE given in position in this plane let fall the perpendicular bB, cC, etc. Take A any fixed point in AE, and let AB, AC, AD, etc. be denoted by a, b, c, etc., Bd, Cc, Dd, etc. by a', b', c', etc., the altitudes at b, c, d, etc. by a'', b'', c'', etc.;



also let the sums of a, b, etc., a', b', etc., a'', b'', etc. be denoted by s, s', s'', and the number of given points by n; finally, let the three coordinates of the point sought, viz. AG, GF, and the altitude above the plane AGF, be denoted by x, y, and z.

Now the square of the distance from F to b is $(x-a)^2+(y-a)^2$, and the difference of the altitudes at F and b is $z-a''$; therefore the square of the first error in distance is $(x-a)^2+(y-a')^2+(z-a'')^2$, the square of the second is $(x-b)^2+(y-b')^2+(z-b'')^2$, the square of the third is $(x-c)+(y-c')^2+z-c'')^2$, etc., etc.

By the preceding theory, the probability of all these errors will be a maximum when the sum of their squares is a minimum; therefore, since each of the three quantities x, y, and z is independent, the three following expressions must each be a minimum, viz.

$$(x-a)^2+(x-b)^2+(x-c)^2+\text{etc.}=\text{min.}$$
$$(y-a')^2+(y-b')^2+(y-c')^2+\text{etc.}=\text{min.}$$
$$(z-a'')^2+(z-b'')^2+(z-c'')^2+\text{etc.}=\text{min.}$$

These three equations put into fluxions and divided by $2dx$, $2dy$, and $2dz$, respectively, become

$$x-a+x-b+x-c+\text{etc.}=0,$$
$$y-a'+y-b'+y-c'+\text{etc.}=0,$$
$$z-a''+z-b''+z-c''+\text{etc.}=0.$$

Whence

$$x=\frac{a+b+c+\cdots}{n},\quad y=\frac{a'+b'+c'+\cdots}{n},\quad z=\frac{a''+b''+c''+\cdots}{n},$$

that is,
$$x=\frac{s}{n},\quad y=\frac{s'}{n},\quad z=\frac{s''}{n}.$$

Whence this rule: divide the sum of each system of ordinates by the number of given points, and the three quotients will be the three orthogonal coordinates of the most probable point required.

From this solution we may deduce the following remarkable consequences.

I. The point sought is so situated that the sum of the errors estimated in any direction whatever is precisely equal to 0.

II. The point sought is precisely in the center of gravity of all the given points, those points being supposed all equal: this is easily shown.

Let p be the mass at any point, and $np=M$ will be the whole mass at all the points. Then we have:

$$x=\frac{a+b+c+\cdots}{n}=\frac{ap+bp+cp+\cdots}{np}=\frac{ap+bp+cp+\cdots}{M}$$

$$y=\frac{a'+b'+c'+\cdots}{n}=\frac{a'p+b'p+c'p+\cdots}{np}$$
$$=\frac{a'p+b'p+c'p+\cdots}{M}$$

$$z=\frac{a''+b''+c''+\cdots}{n}=\frac{a''p+b''p+c''p+\cdots}{np}$$
$$=\frac{a''p+b''p+c''p+\cdots}{M}$$

and these last values of x, y, z, are well known to be the three proper expressions for the three rectangular coordinates of the center of gravity of all the equal quantities of matter p, p, p, etc. Hence it follows that if a point be sought such that the sum of the squares of its distances from any number of fixed points may be a minimum, the point required will be the center of gravity of all the fixed points. Hence also, if a point be sought such that when the squares of its distances from any number of fixed points are multiplied by the fixed numbers p, p', p'', etc., respectively, the sum may be a minimum; the required point will be precisely in the center of gravity of all the fixed points, the quantities of matter at those points being supposed $p, p', p'', \cdots$ respectively.

III. The data of problem II being still supposed; if the locus of a point be required, such, that a point situated anywhere on it may have an equal degree of probability to be the point sought, we may determine the *locus* from the original formulas of the preceding solutions.

We must evidently have the following equation:

$$\left\{\begin{array}{l}(x-a)^2+(x-b)^2+(x-c)^2+\cdots\\ (y-a')^2+(y-b')^2+(y-c')^2+\cdots\\ z-a'')^2+(z-b'')^2+(z-c'')^2+\cdots\end{array}\right\}=nD^2=\text{Constant},$$

that is,

$$\left\{\begin{array}{l}nx^2-2sx+a^2+b^2+c^2+\cdots\\ ny^2-2s'y+a'^2+b'^2+c'^2+\cdots\\ nz^2-2s''z+a''^2+b''^2+c''^2+\cdots\end{array}\right\}=nD^2,$$

which, by putting $a^2+b^2+\cdots+a'^2+b'^2+\cdots+a''^2+b''^2+\cdots=n'D^2$, dividing by n, and putting x', y', z', for the values of

$$\frac{s}{n}, \qquad \frac{s'}{n}, \qquad \frac{s''}{n}$$

becomes

$$\left\{\begin{array}{l}x^2-2x'x\\ y^2-2y'y+D'^2\\ z^2-2z'z\end{array}\right\}=D^2;$$

and this last by completing the squares, transposing D'^2 and putting $D^2-D'^2+x'^2+y'^2+z'^2=r^2$, which is manifestly the equation of a spherical surface having its radius r, and its center in the point of greatest probability.

If therefore the locus of a point be required, such that the sum of the squares of its distances from any number of fixed points may be a constant quantity, the locus sought will be a spherical surface

having its center in the center of gravity of all the fixed points considered as equal to one another.

Hence also, if the *locus* of a point be required, such that when the squares of the distances from any number of fixed points are respectively multiplied by the fixed numbers p, p', p'', etc. the sum of all the products may be a constant quantity, the locus sought will be the surface of a sphere having its center in the common center of gravity of all the points; the quantities of matter in the several points being respectively p, p', p'', etc.

IV. If the *locus* of equal probability (still retaining the data of problem II) be restricted to a given surface, it is clear that the *locus* sought will be the line which is the commmon intersection of the given surface, and of a spherical surface having its center in the point of greatest probability: and the equations of the given surface and of the spherical surface, when referred to the same system of rectangular coordinates, will be the two equations of the *locus* required. If the given surface were plane, or spherical the *locus* of equal probability would be the circumference of a circle.

And therefore, if the locus of a point moving on a given surface be required, such that when the square of its distances from any number of fixed points are multiplied respectively by the numbers p, p', p'', etc., the sum of all the products may be a constant quantity; the *locus* sought will be the common intersection of the given surface, and of a spherical surface having its center in the common center of gravity of all the points, their quantities of matter being supposed to be expressed by the fixed numbers, p, p', p'', etc., respectively.

V. The same data still remaining, we may also determine the points of greatest and least probability on any given line or surface.

Let $V=0$ be the equation of the given line or surface, referred to the same system of coordinates x, y, z. In this case, let the fluxion of the two equations $V=0$ and

$$\left\{\begin{array}{l} (x-a')^2+(x-b)^2+(x-c)^2+\cdots \\ (y-a')^2+(y-b')^2+(y-c')^2+\cdots \\ (z-a'')^2+(z-b'')^2+(z-c'')^2+\cdots \end{array}\right\}=\text{Max. or Min.}$$

be taken; and exterminating anyone of the three fluxions dx, dy, dz, let the coefficients of the other two fluxions in the resulting equation be put each $=0$: this will give two equations from which and the equation $V=0$, the values of x, y, and z may be determined by common algebra.

We may also give a geometrical plan of solution as follows: let a straight line be drawn from the point of greatest probability perpen-

dicular to the given line or surface, and the points of intersection will be those of the maximum or minimum required.

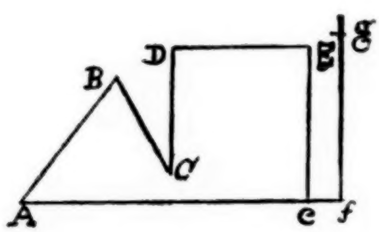
In the very same manner we determine the position of a point on a given line or surface, such that when the squares of its distances from any number of fixed points are respectively multiplied by p, p', p'', etc., the sum of all the products may be a maximum or a minimum.

From the center of gravity of all the fixed points having the quantities of matter p, p', p'', etc. let a straight line be drawn perpendicularly to the given figure, and the intersections will give the points of maxima and minima required.

PROBLEM III.

To correct the Dead Reckoning at Sea, by an observation of the latitude.

Solution: Let $ABCDE$ be a traverse of which the difference of latitude, and departure, according to the dead reckoning are Ae, Ee; the true difference of latitude being Af; and let fg be parallel to eE.



Now the position of the points B, C, D, etc. must be changed in such a manner, that the last point E may fall somewhere on the true parallel fg, and the probability of all these changes must be a maximum.

Let a, b, c, etc., A, B, C, etc. be the lengths and bearings of AB, BC, etc. the radius being unity; x, y, z, etc., X, Y, Z, etc. the motions or translations of the angular points B, C, D, etc. in the directions AB, BC, etc. and in directions perpendicular to the former; D', D'', D''', etc., L', L'', L''', etc., the several changes in departure and latitude, and $L=ef=$ the whole error in latitude.

The several departures are a, sin A, b, sin B, etc. If therfore a and A were variables the fluxion of the departure would evidently be

$$\text{Sin } Ada + a \text{ Cos } AdA,$$

which by putting x and X for da and adA, etc. gives us the equations

$$D' = X \text{ Sin } A + X \text{ Cos } A,$$
$$D'' = y \sin B + Y \text{ Cos } B,$$
$$D''' = z \text{ Sin } C + z \cos C,$$
$$\text{etc.} = \text{etc.}$$

The several differences of latitude are $a \cos A$, $b \cos B$, etc., and because when a, and A are variable the fluxion of $a \cos A$ is

$$\cos Ada - a \sin AdA,$$

therefore putting x and X for da and adA, etc. we have the following equations,

$$L' = x \cos A - X \sin A$$
$$L'' = y \cos B - Y \sin B$$
$$L''' = z \cos C - Z \sin C$$
$$\text{etc.} = \text{etc.}$$

Now the sum of the translations of all the angular points, B, C, D, etc., in the direction ef, must be equal to ef; if therefore we reckon all the bearings one way round from Af, we shall have

I. $x \cos A + y \cos B + \text{etc.} - X \sin A - Y \sin B - \text{etc.} = L$:

and by the preceding theory, if 1 to p be the ratio of the equally probable linear errors in any proposed distance, the former in the direction of the distance, the latter at right angles to the former, we have

II. $$\frac{x^2}{a} + \frac{y^2}{b} + \cdots + \frac{X^2}{p^2a} + \frac{Y^2}{p^2b} + \cdots = \text{min.}$$

Put now the equations I and II into fluxions, and having multiplied the former by m and the latter by $-\frac{1}{2}$, we have, by addition,

$$\left\{ m \cos A - \frac{x}{a} \right\} = dx + \left\{ m \cos B - \frac{y}{b} \right\} dy + \cdots\cdots$$
$$- \left\{ m \sin A + \frac{X}{p^2a} \right\} = dA - \left\{ m \sin B + \frac{Y}{p^2b} \right\} = dY \cdots\cdots = 0$$

This is satisfied by making the several coefficients of the fluxions each $=0$, whence, we obtain the following equations:

$$x = ma \cos A, \qquad y = mb \cos B, \text{ etc.};$$
$$X = -mp^2a \sin A, \qquad Y = -mp^2b \sin B, \text{ etc.}$$

The equations show us that the several motions of the angular points, A, B, C, etc. in the direction of the lengths are directly as the several differences of latitude, and that their motions in directions perpendicular to the former are directly as the departures.

By substituting for x, y, X, Y, etc. their values as just determined we obtain the several corrections in latitude and departure as follows:

$$\begin{array}{ll} L' = amp^2\sin^2 A + am\cos^2 A & D' = (1-p^2)am\sin A\ \mathrm{Cos}\ A \\ L'' = bmp^2\sin^2 B + bm\cos^2 B & D'' = (1-p^2)bm\sin B\ \mathrm{Cos}\ B \\ L''' = cmp^2\sin^2 C + cm\ \mathrm{Cos}^2 C & D''' = (1-p^2)cm\sin C\cos C \\ \text{etc.} = \text{etc.} & \text{etc.} = \text{etc.} \end{array}$$

These equations, by putting $p^2 = 1 + r$, become

$$\begin{array}{ll} L' = am + amr\sin^2 A & D' = amr\sin A\cos A \\ L'' = bm + bmr\sin^2 B & D'' = bmr\sin B\cos B \\ \text{etc.} = \text{etc.} & \text{etc.} = \text{etc.} \end{array}$$

which we may also express more simply thus,

$$\begin{array}{ll} L' = am + \tfrac{1}{2}amr\ \mathrm{versin}\ 2A & D' = \tfrac{1}{2}amr\sin^2 A \\ L'' = bm + \tfrac{1}{2}bmr\ \mathrm{versin}\ 2B & D'' = \tfrac{1}{2}bmr\sin 2B \\ \text{etc.} = \text{etc.} & \text{etc.} = \text{etc.} \end{array}$$

and the value of m is to be derived from the equations

$$m\left\{a+b+\cdots+\tfrac{1}{2}r(a\ \mathrm{versin}\ 2A + b\ \mathrm{versin}\ 2B + \cdots)\right\} = L.$$

But when $p=1$, or $r=0$, which is at once the simplest and most probable value, we have

$$m = \frac{L}{a+b+c+\cdots}$$

also $L' = am$, $L'' = bm$, $L''' = cm$, etc., and $D' = 0$, $D'' = 0$, $D''' = 0$, etc.

These equations furnish the following practical rule for connecting dead reckoning. *Say, as the sum of all the distances in the traverse is to each particular distance, so is the whole errror in latitude to the correction of the latitude corresponding to said distance; those corrections in latitude being always applied in such a manner as to diminish the whole error in latitude; but no corrections whatever must be applied to the several departures by account; and the differences of longitude are to be deduced from the correct differences of latitude and the corresponding departures by dead reckoning.*

But if, as is commonly practiced, the whole departure, the corrections for the particular differences of latitude are unnecessary; and the rule for obtaining the correct difference of longitude is simply this:

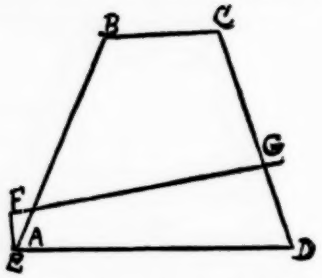
With the departure by account, and the correct difference of latitude find the correct difference of longitude.

If the correct course and distance be required, they must, in like manner, be obtained from the departure by account, and the correct differences of latitude.

From this solution, it appears, that the various rules given by authors for correcting the dead reckoning are totally erroneous; and we hope they will be abandoned by all future writers on the subject.

PROBLEM IV.

To correct a Survey Solution. Let $ABCDE$ be a survey accurately protracted according to the measured lengths and bearings of the sides AB, BC, etc., A the place of beginning, E of ending, AG a meridian, and AF, FE the errors in latitude and departure. Now the problem



requires us to make such changes in the positions of the points B, C, etc. that we may remove the errors AF, FE; in other words, that E may coincide with A, and those changes must be made in the most probable manner. We have therefore to fulfill the three following conditions:

All the changes in departure must remove the error in departure EF.

All the changes in latitude must remove the error in latitude AF.

The probability in all these changes must be a maximum.

Let a, b, c, etc., A, B, C, etc., denote the lengths and bearings of the several sides AB, BC, etc. the radius being unity; x, y, z, etc., X, Y, Z, etc. the exceedingly small motions or translations of the angular points B, C, etc. in the directions of the sides AB, BC, etc. and in direction perpendicular to those lengths; D, L, the errors EF, FA, and D', D'', etc. L', L'', etc. the several corrections required in departure and latitude.

The several departures are a sin A, b sin B, etc. and the fluxion of a sin A being sin Ada cos AdA, by putting x and X for da adA, etc. we have

$$D' = x \sin A + X \cos A$$

$$D'' = y \sin B + Y \cos B$$

$$\text{etc.} = \text{etc.},$$

and therefore

$$\text{I.} \quad x \sin A + y \sin B + \text{etc.} + X \cos A + Y \cos B + \text{etc.} = D.$$

Again, the differences of latitude are a cos A, b cos B, etc. and because the fluxion of a cos A is

$$da \cos A - a \sin A\, dA,$$

by putting x and X for da and $a\ d\ A$, etc. we have

$$L' = x \cos A - X \sin A$$

$$L'' = y \cos B - Y \sin B$$

$$\text{etc.} = \text{etc.}$$

and therefore, reckoning the bearings all one way round from the meridian AG, we have

$$\text{II.} \quad X \operatorname{Cos} A + Y \operatorname{Cos} B + \cdots - X \sin A - Y \sin B \cdots = Z.$$

Also, retaining the same signification of the latter p, we have, by the preceding theory, for the greatest probability,

$$\text{III.} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \cdots + \frac{X^2}{p^2 a} + \frac{Y^2}{p^2 b} + \cdots = \text{min.}$$

Now let the fluxions of the three equations I, II, III be multiplied, respectively, by m, n, $-\frac{1}{2}$, and by addition we have

$$\left\{\begin{array}{l} m \sin A\ dx + m \sin B\ dy + \cdots + m \operatorname{Cos} A\ dX + m \operatorname{Cos} B dY + \cdots \\ n \operatorname{Cos} A\ dx + n \operatorname{Cos} B\ dy + \cdots - n \sin A dX - n \sin B dY - \cdots \\ \dfrac{x}{a} dx - \dfrac{y}{b} dy - \cdots - \dfrac{X}{p^2 a} dX - \dfrac{Y}{p^2 b} dY - = 0. \end{array}\right.$$

This equation is satisfied by making the sum of the coefficients of each

fluxion separately equal to 0. When we obtain the following equation

$$x = ma \sin A + na \operatorname{Cos} A$$

$$y = mb \sin B + nb \cos B$$

$$\text{etc.} = \text{etc.}$$

$$X = am^2 p^2 \cos A - np^2 \sin A$$

$$Y = bmp^2 \cos B - np^2 \sin B$$

$$\text{etc.} = \text{etc.}$$

If the departures and differences of latitude, etc. be denoted by a', b', c', etc. a'', b'', c'', etc. we may express the several values of $x, y, \cdots, X, Y, \cdots$ thus,

$$x = m\,a' + n\,a'' \qquad X = p^2(m\,a'' - n\,a')$$

$$y = m\,b' + n\,b'', \qquad Y = p^2(m\,b'' - n\,b')$$

$$z = m\,c' + n\,c'', \qquad Z = p^2(mc'' - n\,c')$$

$$\text{etc.} = \text{etc.} \qquad \text{etc.} = \text{etc.}$$

But the proper algebraic signs of sin A, Cos A, sin B, Cos B, etc. must also be transferred to those values of $a', b', \cdots, a'', B'', \cdots$

Putting now for $x, y, \cdots, X, Y, \cdots$ their values, we obtain the several corrections in departure and latitude as follows:

$$D' = am \sin^2 A + am\,p^2 \cos^2 A + (a\,m - a\,n\,p^2) \sin A \cos A.$$

$$D'' = b\,m^2 \sin^2 B + b\,m\,p^2 \cos^2 B + (b\,n - b\,n\,p^2) \sin B \cos B$$

$$\text{etc.} = \text{etc.}$$

$$L'a\,n \cos^2 A + a\,n\,p^2 \sin{}^2 A + (am - a\,m\,p^2) \sin A \cos A,$$

$$L'' = b\,n \cos^2 B + b\,n\,p^2 \sin^2 B + (b\,m - b\,m\,p^2) \sin B \cos B,$$

$$\text{etc.} = \text{etc.}$$

which expressions, by putting $p^2 = 1 + r$, become

$$D' = a\,m + am\,r \cos^2 A - a\,n\,r \sin A \cos A,$$

$$D'' = b\,m + b\,m\,r \cos^2 B - b\,n\,r \sin B \cos B,$$

$$\text{etc.} = \text{etc.}$$

$$L' = a\,n + an\,r \sin^2 A - a\,mr \sin A \cos A,$$

$$L'' = b\,n + b\,n\,r \sin^2 A - b\,m\,r \sin B \cos B,$$

$$\text{etc.} = \text{etc.}$$

and these may be transformed into the following:

$$D' = am + \frac{ar}{2}(m + m \cos 2A - n \sin 2A) \ ,$$

$$D'' = bm + \frac{br}{2}(m + m \cos^2 B - n \sin^2 B),$$
$$\text{etc.} = \text{etc.}$$

$$L' = an + \frac{ar}{2}(n - n \cos^2 A - m \sin^2 A),$$

$$L'' = bn \frac{br}{2}(n - n \cos^2 B - m \sin^2 B),$$

$$\text{etc.} = \text{etc.}$$

The values of m and n are discovered by equating the sum of the values of D', D'', etc. to D, and the sum of those of L', L'', etc., to L; m and n will therefore be found by simple equations.

In these calculations the sign of all angles in the third and fourth quandrants must be taken negatively, as well as the cosines of those in the second and third quandrants; according to the common rule in such cases.

Also when the errors EF, AF are on the same side of the beginning, A, with the first departure and difference of latitude, then D and L must be taken negatively.

The simplest case of the problem is, when $p = 1$, or $r = 0$, which is also the most probable supposition by Problem I; besides, this seems to agree best with the imperfections of the common instruments used in surveying.

In this case, the values of m, n, and of the required corrections in departure and latitude are as follows:

$$m = \frac{D}{a+b+c+\cdots}, \qquad n = \frac{L}{a+b+c+\cdots}$$

$$D' = m\,a, \quad D'' = mb, \quad D''' = mc, \text{ etc.}$$

$$L' = n\,a, \quad L'' = n\,b, \quad L''' = nc, \text{ etc.}$$

Hence, the following practical rule for correcting a survey: *Say, as the sum of all the distances is to each particular distance*, so is the whole *error in departure to the correction of the corresponding departure; each*

correction being so applied as to diminish the whole error in departure; proceed in the same way for the corrections in latitude.

When $p=1$, as in the practical rule, the motions of the angular points B, C, D, etc. are parallel to the whole linear error EA. This appears by imagining the meridian to coincide with EA; for in this case $D=0$, and therefore $D=0$, $D''=0$, $D'''=0$, etc. which equations show that the motions of B, C, D, etc. are parallel to EA. Also because $L'=na$, $L''=nb$, $L'''=nc$, etc., it follows that the motions of B, C, D, etc., in the direction EA are proportional to the several distances a, b, c, etc.

But when p is not equal to unity, the motions of B'' C'', etc. are not parallel to EA: when $p=0$, their motions are in the directions a, b, c, etc. because in this case the equations

$X=p^2(ma''-na')$, $Y=p^2(mb''-nb')$, etc; become $X=0$ $Y=0$, etc.

When p is infinite, we have only to remove p^2 from the values of X, Y, etc., to those of x, y, etc., and make $p=0$; in this case the equations

$$x=p^2(ma'+na''), \quad y=p^2(mb'+mb''), \text{ etc.}$$

become $\quad x=0, \quad y=0,$

and the remaining motions $X=ma''-na'$, $Y=mb''-mb'$, etc., are manifestly perpendicular to the distances a, b, c, etc.

From this investigation, it appears that the rule hitherto given by authors for correcting a survey are altogether erroneous, and ought to be entirely rejected, the true method here given is exemplified by Mr. Bowditch, in his solution of Mr. Patterson's question of correcting a survey; his practical rule and mine being precisely the same.

I have applied the principle of this essay to the determination of the most probable value of the earth's ellipticity, etc. but want of room will not permit me to give the investigations at this time."

Thus ends the most important contribution to any of earlier mathematical journals of America.

The editor states that he received two solutions of Mr. Garrett's prize problem, one from Mr. Garrett himself and the other from Dr. Bowditch. Because of the importance of the problem and the improvements to be made in the next publication of the *Nautical Almanac*, by Mr. Garrett, the editor decided to postpone the awarding of the prize till the next number. In the new number of the *Nautical Almanac*, (1809) the moon's right ascension was to be given in min. and sec. instead of min., only, as previously and it was the request of the editor that those contending for the prize must show how their solutions may

be applied to the case in which the moon's right ascension is given to the nearest second of a degree.

The question is then reproduced. It reads as follows:

The sun's right ascension at noon, and the moon's right ascension at noon, and midnight, being always given in the *Nautical Almanac* for Greenwich; required, on any day, the time when the moon's center will be on the meridian of any place whose longitude from Greenwich is known. For example, at Philadelphia, the 3rd day of May, 1809, Philadelphia being supposed 5 hours, 0 minutes, 55 seconds west from Greenwich.

Mr. Garrett, the proposer, offers $6.00 for the best and most accurate solution. No solution of Mr. Patterson's prize problem having been received by the editor it was also reproposed. The problem is,

To find the most simple and accurate method of finding the variation of the magnetic needle on land without the aid of any other instrument except the common surveying compass, and a watch that will keep time within five minutes in the week, and the greatest error in variation not to exceed five minutes of a degree. The method must, of course, have directions for correcting the watch within the necessary degree of accuracy.

The editor announces, page 110, that he received from Dr. Bowditch, a correct solution of the problem of the *Elastic Oval*. Dr. Bowditch's solution was general and, therfore, capable of being applied to more complex cases. In order to call forth different methods of solution, the editor proposed the two following problems:

(a) It is required to investigate the nature of the *Elastic Oval*, or the figure which a perfectly elastic circular hoop, of uniform strength and thickness will assume, when acted upon by two equal and opposite forces at the extremities of a diameter.

(b) It is required to investigate the nature of the Ellipsis Elastica Volvens, or the figure which a perfectly elastic circular hoop, of uniform strength and thickness, and density, will assume when it revolves with uniform angular velocity about one of the diameters as an axis, in free and non-gravitating space.

Dr. Bowditch, on page 111, raised the question as to whether the corrections of the logarithms of the radius vector of the earth's orbit, arising from the disturbing force of Jupiter is rightly allowed for in Table XVI of the 3rd Edition of Lelande's Astronomy.

This number, No. 4, of the *Analyst*, concludes with sixteen problems proposed for solution, making a total of 46. The problems are

numbered with two numbers,—one a Roman numeral placed in front of the word, "Question" indicates the order in the Number of the *Analyst*, the second, an Arabic numeral, placed after the word "Question", indicating its order from the beginning of the *Analyst*.

The last problem in No. 4 is a prize problem, proposed by John D. Craig, Baltimore, Md. The amount of the prize is not stated. The problem reads as follows:

On examining the font of an old Cathedral, its top and bottom diameters were found to be $28^4/_5$ and $9^3/_5$ in. respectively; the concavity or bason was a hemisphere, 24 in. in diameter and the thickness of the marble was everywhere such that if a straight line were any how drawn through the center of said hemisphere the part thereof intercepted by the internal and external surfaces would be everywhere as the secant of that line's depression. It is required thence to determine the height and solid capacity of the Font.

With this number, (4), it appears, the publication of the *Analyst* was suspended for five years.

The fifth number of the *Analyst* bears the date, March 1st, 1814. The only copy of this number known to us is in the Astor Library, New York City. It is bound up with some other scientific tracts, among which are six numbers of the *Mathematical Diary*. No mention is made in this number about the suspension of publication and from the title page, one would infer that this number was really the beginning of this journal. It is stated that number II will appear in June.

The following is the contents of the title page:

THE ANALYST

No. I.

Containing

New Elucidations, Improvements, and Discoveries,
in the Various Branches of the
MATHEMATICS
with
Solutions of New and Interesting Questions
Proposed and Resolved by
INGENUOUS CORRESPONDENTS

Conducted by

R. ADRAIN, A. M.

Fellow of the Am. Phil. Society, held at Phil., of the American Academy

of Arts and Sciences, of the Literary and Philosophical Society of N. Y., and Professor of Mathematics and Natural Philosophy
in Columbia College, N. Y.

Vol. I.

Utile Dulci

——o——

New York.
Printed and Published by Geo. Long
No. 71 Pearl Street
March 1st, 1814.

The preface of this number reads as follows:

PREFACE

The advantages resulting from such a work as is contemplated in the plan of the *Analyst* are too obvious to be formally insisted on, or to require a tedious enumeration: the most judicious Mathematicians of the last two centuries have concurred in their sentiments, respecting the utility of similar publications; and experience has confirmed their judgment in the most satisfactory manner.

Of this advantage, one, by no means, the least important, consists in bringing into deserved notice the talents of many ingenuous persons, who, from their situations, have few opportunities of being exhibited in a just point of view, or of having their abilities properly employed for the public good. Justly impressed by this consideration, the conductor of the *Analyst* will take every opportunity of displaying, in fair and honorable manner, such problems, solutions, or essays, as are truly indicative of mathematical genius.

It will be proper at the same time to assign a suitable portion of the *Analyst* to problems within the reach of the studious though less experienced contributors; and thus to conduct them by easy and natural steps to those more elevated stations, from which are beheld, with admiration and delight, the never fading prospects of intellectual and physical nature adorned with the elegance of taste, and illumined by the beams of science.

The improvements of science, and its new applications to philosophy and the arts, are also primary objects in the scheme of this publication. Even the few numbers of our former series of the *Analyst* contained several valuable improvements, which have since been transferred to the best standard works on Navigation and Surveying. We may reasonably hope to obtain results of similar utility, in the course

of the present publication. All communications of such improvements will receive the greatest attention from the conductor, and be faithfully laid before the readers of the *Analyst*.

In a word, the objects of the work are to accelerate the progress of the young, excite to action the power of genius, display in a proper light those talents which should be known to the public, and afford a place for such improvements and discoveries as may be useful to mankind."

The first article is by the editor, Robert Adrian, on, "Algebraic Method of Demonstrating the Propositions in the Fifth Book of Euclid's Elements According to the Text and Arrangement in Simpson's Edition, Adopted to the Instruction of Youth in Schools and Colleges."

Professor Adrian says, ... the fifth book of Euclid still presents great difficulties to learners, and is, in general, less understood than any other part of the elements of Geometry. The present essay is intended to remove these difficulties, and consequently to enable learners to understand in a sufficient degree the doctrine of proportion, previously to their entering on the sixth book of Euclid, in which that doctrine is indispensible."

The author, instead of using Euclid's definition of proportion as given in the 5th definition of the 5th book, makes use of the common algebraic definition, and then shows the equivalence of the two. The author says, "This perfect agreement between the two definitions is a matter of great importance in the doctrine of proportion, and has not (as far as I can learn) been discovered by any preceding mathematician." The second article, so called, consists of sixteen problems proposed for solution, with which this number of the *Analyst* concludes.

The sixteenth problem is a prize problem proposed by Robert Adrain. The problem requires to find the height of the mouth of the Mississippi River above its source, the elevation of its source above the level of the sea being given by the best authorities and the mean ellipticity of the earth being

$$\frac{1}{320}.$$

Problem 15, proposed by Dr. Bowditch, reads as follows:

Suppose a stick of timber, as a cylinder, parallelopiped, etc. the length of which is much greater than the width, or thickness, to rest horizontally on supports at each end, and that weights are gradually applied to the middle point till the timber breaks, the weight then applied being W; the vertical distance passed over by the middle

point at the moment of breaking being a; suppose also that the weight w, which would gradually bend it through the distance s, is to W as s to a, (which is the usual law of elastic bodies). Upon this principle it is required to find the weight W which applied all at once at the same middle point would have just force enough to break the same timber, supposing that it is only necessary to bend the middle point through the space a to produce the effect, neglecting in the calculation, the weight of the timber.

Problem 13, proposed by J. Roosevelt of New York. It reads as follows:

In an upright cone with elliptic base, given the altitude and the axes of the base, it is required to find the position of a circular section of the cone.

Thus we see that a Roosevelt of a century ago proposed mathematical nuts for the mathematicians of his day while the Roosevelt of the present day proposes political nuts for the politicians of his day (1909).

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Foot-Notes to the Chapter on "Theory of Equations"

By EMORY P. STARKE

1. *Division and the Remainder Theorem.* Much discussion has centered about the proof usually given for the remainder theorem, viz: Divide $f(x)$, a rational integral function of degree n, by $x-c$ and obtain the identity

$$f(x) \equiv Q(x) \cdot (x-c) + R, \tag{1}$$

where $Q(x)$, the quotient, is a polynomial of degree $n-1$ and R, the remainder, is a constant. In (1) put $x=c$ to find

$$f(r) = Q(c) \cdot 0 + R = R \tag{2}$$

which completes the proof.

This form of proof is carefully avoided by many teachers and authors who argue that (1) is not valid when the divisor $x-c$ is zero, hence we cannot use in (2) the very value for x needed to complete the proof. The fact that instructors disagree here suggests that great care is needed in presentation if students are not be confused and misled.

Neither the objection to the proof nor the explanations usually heard are valid. The difficulty arises because we fail to recognize that we deal, even in elementary algebra, with two distinct sets of quantities: (a) the *field* of complex constants from which are taken the coefficients of $f(x)$ and the values which may be substituted for x, and (b) the *ring* of rational integral functions, $f(x)$. In (a) the operation of multiplication has an inverse, called division: every element, e, (except 0) of the field has an inverse, $1/e$; there exists no inverse of 0. In (b) the inverse of multiplication is not defined; the name "division" is used to represent an operation of repeated subtraction.* The funda-

*This distinction between the two concepts of division in form for elementary presentation is given in detail in Fine, *A College Algebra*, p. 27 ff.

mental theorem here is: given any two elements $f(x)$ and $d(x)$, with $d(x)$ not identically 0, there exists a unique pair of elements $Q(x)$ and $R(x)$, with the degree of $R(x)$ lower than that of $d(x)$, such that, identically,

$$f(x) \equiv Q(x) \cdot d(x) + R(x). \tag{3}$$

By "identically" we mean that if the operations indicated in the right member are carried out, the left member is obtained, whatever the value of x. The proof of the theorem is implicit in the multiplications and subtractions involved in the usual process (algorithm) of division of polynomials.

While one would not wish to mention general rings and fields in a freshman class, the concept and proof of (3) should be mastered before being used in (1).

2. *Proofs by Mathematical Induction.* In most college courses in algebra, mathematical induction is treated before the theory of equations. Both topics suffer if the induction inherent in at least three theorems on equations is not exploited. (There seems no excuse for limiting induction to the proof of the binomial theorem.) It is a revelation to observe the improved response of students who, having previously studied the text presentation, are then shown the induction point of view.

Suppose that the remainder theorem and its corollary, the factor theorem, have been proved and that the fundamental theorem of algebra has been postulated. We desire to prove: if $f(x)$ *is of degree* n, *then* $f(x)=0$ *has just* n *roots, each multiple root being counted according to its order.* The theorem is known to hold for the linear and the quadratic cases, i. e. for $n=1,2$. Assume that it holds for $n=1,2,\cdots,k$ and consider a polynomial $f(x)$ of degree $k+1$. $f(x)=0$ has a root (by the fundamental theorem) which we may call c. Then (by the factor theorem) $f(x)\equiv(x-c)Q(x)=0$, where $Q(x)$ is of degree k. But (by the hypothesis on k) $Q(x)=0$ has k roots. Thus $f(x)=0$ has these k roots and the root c, i. e. has $k+1$ roots. (Even if c is a duplicate of one of the k roots, it will be a multiple root of $f(x)=0$ of order one higher; so that $f(x)=0$ has $k+1$ roots.) The induction is thus complete.

In the proof of Descartes' rule, the induction is thinly disguised—why should it not be presented frankly? The first step here becomes trivial: if $f(x)=0$ has 0 positive roots, it is obvious that 0 cannot exceed the number of variations in sign. Let $f(x)=0$ have $p, (p \geqq 0)$, positive roots and v, $(v \geqq p)$, variations in sign; then for any r, $(r>0)$, $(x-r)f(x)=0$ has $p+1$ positive roots, and any of the usual text proofs

shows directly that there are at least $v+1$ variations in sign. Since $v+1 \geqq p+1$, the induction is complete. A more useful form of Descartes' rule results when, instead of the trivial first step above, we show: v and p are alike even or alike odd. Thus, let

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n.$$

From graphical considerations, if a_0 and a_n have like signs there are no positive roots or there is an even number of them: if unlike, an odd number. But if a_0 and a_n have like signs, there are evidently no variations in sign or an even number of them: if unlike, an odd number. This, together with the proof about $(x-r)f(x)=0$, shows that $v-p$ is always a positive even integer or zero.

The theorem on relations of roots to coefficients is given by induction in many books.

3. *Horner's and Newton's Methods.* In his study of Horner's method, the student is frequently more impressed by the successive diminutions of the roots (which comprise the bulk of the mechanical work) than by the more important consideration of how to determine the amounts of the various diminutions. The usual method of estimating the next figure at any stage of the work is to solve the last transformed equation after neglecting powers of x higher than the first. This is equivalent to Newton's method, as shown below. Interpolation is preferred for this step, as offering several advantages: (1) being already familiar in connection with the use of tables, (2) having a simple graphical interpretation independent of the derivative and (3) providing a check on numerical accuracy.

A concrete example will sufficiently illustrate these points.

$$f(x) \equiv x^3 + x^2 - 18 = 0$$

has a root r between 2 and 3. Determine $f(2) = -6$, $f(3) = 18$. If $f(r) = 0$, we may approximate r by interpolation, thus:

$$(r-2)/(3-2) = (0 - -\bar{6})/(18 - -\bar{6})$$

or $r = 2+6/24$ or $r = 2.2+$. Next, diminish the roots of $f(x) = 0$ by 2: with $x_1 = x-2$ or $x = x_1+2$, $f(x) \equiv f(x_1+2)$ becomes

$$f_1(x_1) \equiv x_1^3 + 7x_1^2 + 16x_1 - 6 = 0,$$

which has, as we already know, a root approximately $r_1 = .2$ (from $r = 2.2$ and $r_1 = r-2$). In passing, we note that $x_1 = 1$ corresponds to $x = 3$, and thus $f_1(1)$ should equal $f(3) = 18$: the determination that $f_1(1)$ is 18 is a check on the previous numerical work. For the next section of the work, we find $f_1(.2) = -2.512$, $f_1(.3) = -.543$. Evi-

dently we need also $f_1(.4)=1.584$ to determine that r_1 is actually between .3 and .4; interpolation gives $r_1=.32+$. With $x_2=x_1-.3$, $f_1(x_1)$ becomes $f_2(x_2)\equiv x_2^3+7.9x_2^2+20.47x_2-.543=0$ with a root r_2 approximately .02. We check these last steps by showing that $f_2(.1)=f_1(.4)=1.584$ and proceed by analogous steps for further decimal places.

In the above we determined expansions of the form $f(x+h)$, usually by synthetic division. These expansions are the same as those obtained by Taylor's theorem,

$$f(x+h)\equiv f^{(n)}(h)\cdot x^n/n!+\cdots+f''(h)\cdot x^2/2!+f'(h)\cdot x+f(h),$$

where the coefficients involve the values of the several derivatives of $f(x)$ for $x=h$. If we neglect powers of x higher than the first in $f(x+h)=0$, there results $x=-f(h)/f'(h)$. This is the basic formula of Newton's method: if h is an approximation to a root of $f(x)=0$, then $x+h=h-f(h)/f'(h)$ is, in general, a better approximation. Step by step the approximation is thus improved until any required degree of accuracy is obtained. Ordinarily we employ only the first significant figure of the correction $-f(h)/f'(h)$ and then diminish the roots of $f(x)=0$ by this amount, using synthetic division rather than Taylor's theorem. Thus the distinction between Horner's and Newton's methods becomes very slight. The distinction of real significance is between the two methods of obtaining the next approximation, viz. (a) interpolation and (b) the formula $x_{i+1}=x_i-f(x_i)/f'(x_i)$. Either of these methods (without the process of diminishing the roots by synthetic division) will yield solutions of transcendental equations.

Both methods have simple graphical interpretations. (a) The line joining $(a, f(a))$ and $(b, f(b))$ crosses the X-axis at the point given by the interpolation, $(x-a)/(b-a)=(0-f(a))/(f(b)-f(a))$; (b) the line through $(h, f(h))$ tangent to the curve $y=f(x)$ crosses the X-axis at the point $x=h-f(h)/f'(h)$. Thus (a) gives the approximation resulting from replacing the curve by a straight line joining two nearby points, while (b) is the result of using the tangent to the curve at a point. Thus (b) is the limit of (a) as $b\to h$ and $a\to h$.

It appears that the process of diminishing the roots should not be over-emphasized. If only one or two decimal places are required (or if a calculating machine is available), results are obtained satisfactorily by interpolation or by Newton's approximation formula alone.

Mathematical World News

Edited by
L. J. Adams

Professor Julian L. Coolidge, senior member of the department of mathematics at Harvard University, will retire on September 1, 1940, with the title Professor of Mathematics Emeritus and Master of Lowell House Emeritus. Professor Coolidge has been a member of the department of mathematics for forty years and Master of Lowell House for ten years.

A note by Professor G. A. Miller, University of Illinois, is in the March 22, 1940 issue of *Science*. In the note Professor Miller calls attention to the differences between volume I of the Encyklopädie der Mathematischen Wissenchaften as published originally in 1898 and as published in revised form since the summer of 1939. He also cites other instances of the changes in mathematics as can be observed by comparing editions of various encyclopedias.

The following are advanced courses to be offered by the department of mathematics during the 1940 Summer Session at Texas Technological College:

First Term, June 6 to July 15

1. *Differential Equations*. Professor Michie.
2. *Vector Analysis*. Dr. Ollmann.
3. *Research and Reading Course for M. A. Thesis*. Dr. Hazlewood.

Second Term, July 16 to August 23.

1. *Advanced Calculus*. Professor Michie.
2. *Complex Variable*. Professor Heineman.
3. *Research and Reading Course for M. A. Thesis*. Professor Michie.

The usual elementary courses in algebra, trigonometry, analytic geometry, and beginners' calculus will be offered.

Three years ago the Mathematics Club was organized at Alfred University in Alfred, New York. The officers this year are: President, Esther T. Gent; Secretary, Mildred Haerter; Treasurer, Beth Olsguivy; Faculty Advisor, Dr. L. L. Lowenstein. A constitution was adopted by a unanimous vote on February 13, 1940. Article V provides for an honorary branch of the Mathematics Club. Juniors and seniors who

are outstanding and who receive an honor index in mathematics are eligible. The name Pi Delta Mu was chosen from the Greek translation of the motto "Let us advance through mathematics." Article VI provides for some sort of an award to be given to the senior student who is most outstanding in the study of mathematics. This award will be given at Commencement. Last year the club purchased several books for the college library. They plan to do so again this year. The program for this year included: 1. *Mathematical Puzzles,* Professor W. V. Nevins, Alfred University. 2. *Magic Squares,* Dr. Walker, Cornell University. 3. *The set of Algebraic Numbers is a Countable set,* Ward E. Fax, senior in mathematics, Alfred University. 4. *The Nine-Point Circle,* Esther T. Gent, senior in mathematics, Alfred University. 5. *Imaginary Numbers are Real,* Dr. L. L. Lowenstein, Alfred University. Dr. H. M. Gehman of the University of Buffalo will speak at the April meeting and Dr. H. T. R. Aude of Colgate University will speak at the May meeting.

Professor W. M. Whyburn, chairman of the department of mathematics of the University of California at Los Angeles, will be a member of the summer school faculty of the University of Virginia during the coming summer. For the second session Professor Whyburn will offer graduate courses in *Foundations of Geometry* and *Real Variable.*

Professor W. L. Hart, University of Minnesota, is on sabbatical leave for the academic year 1939-40. Professor Hart is residing in Santa Monica, California, and is in close contact with the University of California at Los Angeles and the California Institute of Technology.

The annual meeting of the Northern California Section of the Mathematical Association of America was held on Saturday, January 27th, 1940 at the University of California, Berkeley. The program included:

1. *Simple Mathematical Theory of Economic Relief.* Professor G. C. Evans, University of California.
2. *Mathematics and the Constructive Arts.* Professor W. F. Durand, Stanford University.
3. *Geometric Representation of Certain Magnetic Fields.* Professor F. R. Morris, Fresno State College.
4. *Shall We Defer the Teaching of Algebra to the Tenth Grade?* Miss Harriet Welch, Lowell High School, San Francisco.
5. *Some Difficulties with Mathematics in a Core Curriculum.* Dr. Vern James, Menlo Junior College.
6. A General Solution of $x_1^2+x_2^2+\cdots+x_n^2=m^2$. Mr. A. L. McCarty, San Francisco Junior College.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 273. Proposed by *M. S. Robertson*, Rutgers University.

Given the two real circles cutting orthogonally:

$$x^2+y^2+Dx+Ey+F=0, \quad x^2+y^2=F, \quad F>0,$$

and a straight line $Ax+By+C=0$, $C\neq 0$.

Show that

$$C^2(x^2+y^2+Dx+Ey+F)-F(Ax+By+C)^2=0$$

represents two distinct straight lines when, and only when, one or the other of the two following conditions holds:

(a) the given line is the radical axis of the two circles, or
(b) the given straight line is tangent to the second circle at a point of intersection of the two given circles.

Solution by *Johannes Mahrenholz*, Cottbus, Germany.

The equation $C^2(x^2+y^2+Dx+Ey+F)-F(Ax+By+C)^2=0$ represents two distinct straight lines if and only if its "discriminant" vanishes, i. e.

$$(CD-2AF)^2+(CE-2BF)^2-F(AE-BD)^2=0,$$

from which the factor $-C^4\neq 0$ has been removed.
This expression is an equality when

(a) $A=D, \quad B=E, \quad C=2F;$

(b) $A=(-2DF\pm E\sqrt{(D^2+E^2)F-4F^2})/(D^2+E^2),$

$B=(-2EF\mp D\sqrt{(D^2+E^2)F-4F^2})/(D^2+E^2), \quad C=-F.$

That these are the two cases specified in the problem is easily seen from the following:

(a) $Dx+Ey+2F=0$ is the radical axis of the two circles;

(b) (A, B) is a point of intersection of the two given circles and hence $Ax+By=F$ is the tangent to the second circle at (A, B).

The *Proposer* notes that in all other cases where the given straight line cuts the first circle in L and M, the locus is an ellipse, parabola or hyperbola whose axis is the right bisector of the chord LM. The locus is an ellipse if the line does not meet the second circle; a parabola if it touches the second circle; a hyperbola if it cuts both circles. In all cases the locus passes through L, M and the origin.

Editor's Note. It is instructive to subject the problem to a different analysis. Let the two circles intersect in points P and Q, and let L and M be the intersections (not necessarily real or distinct) of the line with the first circle. By inspection, the conic is seen to pass through the origin and through L and M, and in fact is tangent to the first circle at L and M, as may be seen from the derivatives of the two curves at these points. If now the conic degenerates into two distinct straight lines, they must be the tangents to the first circle at L and M; but this is not possible, if the conic is to contain the origin, unless L or M coincides with P or Q. Conversely, if $Ax+By+C=0$ passes through P or Q the conic must be degenerate: this follows since the line joining P (or Q) to the origin is tangent to the first circle, and a non-degenerate conic cannot have a second point in common with one of its tangents. Thus the conditions of the problem hold for *any* line through P or Q, and not merely for the lines (a) and (b).

No. 315. Proposed by *H. S. Grant*, Rutgers University.

$F(x, y)=0$ and $F(\rho \cos \theta, \rho \sin \theta)=0$ when solved for y, x, and ρ explicitly define single-valued, continuous functions of x, y, and θ, respectively. Consider

$$A_1=\left|\int_{x_1}^{x_2} y\,dx\right|;\quad A_2=\left|\int_{y_1}^{y_2} x\,dy\right|;\quad A_3=\left|\int_{\alpha}^{\beta} \rho^2 d\theta\right|$$

where (x_1, y_1) and (x_2, y_2) are the rectangular coordinates of any two points on the curve and $\alpha=\text{arc}\tan(y_1/x_1)$, $\beta=\text{arc}\tan(y_2/x_2)$. For what points is (1) $A_1=A_2$; (2) $A_1=A_3$; (3) $A_2=A_3$; (4) $A_1=A_2=A_3$?

Solution by the *Proposer*.

If the problem is to have meaning, the functions must be differentiable as well as continuous.

$$(1) \qquad \int_{x_1}^{x_2} y\,dx = \int_{y_1}^{y_2} y(dx/dy)dy.$$

The single-valuedness of both y and x implies that each function is monotone. Thus for $A_1 = A_2$, it is necessary and sufficient that $y(dx/dy) = \pm x$, giving the lines $y = kx$ and the rectangular hyperbolas $xy = k$.

(2) The relations $x = \rho \cos\theta$; $y = \rho \sin\theta$; $\alpha = \text{arc} \tan(y_1/x_1)$; $\beta = \text{arc} \tan(y_2/x_2)$ reduce

$$\int_{x_1}^{x_2} y\,dx \quad \text{to} \quad \int_{\alpha}^{\beta} \rho \sin\theta \left[\cos\theta \frac{d\rho}{d\theta} - \rho \sin\theta\right] d\theta.$$

Hence for $A_1 = A_2$, it is necessary and sufficient that

$$\sin 2\theta(d\rho/d\theta) - 2\rho \sin^2\theta = \pm\rho, \quad \rho \neq 0.$$

This gives $d\rho/\rho = \tan\theta\, d\theta \pm \frac{1}{2}\sec^2\theta d\theta/\tan\theta$, whence we have the cubical parabolas $y = kx^3$ and the rectangular hyperbolas $xy = k$.

(3) Proceeding as in (2), using

$$\int_{y_1}^{y_2} x\,dy,$$

we obtain the cubical parabolas $x = ky^3$ and the rectangular hyperbolas $xy = k$.

(4) From the preceding results we conclude that the only non-trivial functions for which $A_1 = A_2 = A_3$ are the rectangular hyperbolas $xy = k$.

No. 323. Proposed by *W. C. Janes*, Lincoln, Nebraska.

Verify the following identities:*

$$(1) \quad 2^{2n-1}\sum_{r=0}^{n-1} \frac{(2n-r-1)\cdots(2n-2r)}{r!} = \sum_{r=0}^{n-1} \frac{2n\cdots(2n-2r)}{(2r+1)!}\,5^r\ ;$$

*It was pointed out in this Magazine, January number, p. 225 that the first terms in the summations of the left members of (1) and (2), corresponding to $r = 0$, are to be assigned the value 1.

$$(2)\qquad 2^{2n}\sum_{r=0}^{n-1}\frac{(2n-r)\cdots(2n-2r+1)}{r!}$$

$$=\sum_{r=0}^{n-1}\frac{(2n+1)\cdots(2n-2r+1)}{(2r+1)!}5^r.$$

Solution by the *Editors.*

Let us put

$$a_{n+1}=\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots,$$

$$b_n=\binom{n}{1}+5\binom{n}{3}+5^2\binom{n}{5}+\cdots=\frac{(\sqrt{5}+1)^n-(1-\sqrt{5})^n}{2\sqrt{5}},$$

$$c_n=\binom{n}{0}+5\binom{n}{2}+5^2\binom{n}{4}+\cdots=\frac{(\sqrt{5}+1)^n+(1-\sqrt{5})^n}{2}.$$

We have $a_1=1$, $a_2=1$, $a_3=2$, $a_4=3$; $b_1=1$, $b_2=2$, $b_3=8$, $b_4=24$.

The known relation $\binom{k}{r}=\binom{k-1}{r}+\binom{k-1}{r-1}$

applied to the above sums gives

$$(3)\qquad a_{n+1}=a_n+a_{n-1},$$

$$(4)\qquad b_n=b_{n-1}+c_{n-1},\qquad c_n=5b_{n-1}+c_{n-1}.$$

From (3) we see that the a_i are the terms of the Fibonacci series (see American Mathematical Monthly, 1918, pp. 232-238).

From (4) we eliminate c_{n-1} and c_n to obtain

$$(5)\qquad b_{n+1}=2b_n+4b_{n-1}.$$

After noting that $b_n=2^{n-1}a_n$ is true for $n=1, 2, 3, 4$, assume its truth for $n=1, 2, \cdots, k$. Then by (5) and (3)

$$b_{k+1}=2b_k+4b_{k-1}=2\cdot 2^{k-1}a_k+4\cdot 2^{k-2}a_{k-1}=2^k(a_k+a_{k-1})=2^ka_{k+1}.$$

Hence by induction $b_n=2^{n-1}a_n$ is true for all values. (1) and (2) may be put $2^{2n-1}a_{2n}=b_{2n}$, $2^{2n}a_{2n+1}=b_{2n+1}$, in which form their truth has just been demonstrated.

No. 329. Proposed by *Robert C. Yates*, Louisiana State University.

P is a point in the plane of a given circle C. With compasses alone find the intersection of the circle and its diameter through P.

Solution by *B. A. Hausmann*, S. J., University of Detroit, Detroit, Mich.

With P as center and a convenient radius, draw an arc to intersect the given circle whose center is C in points A and B. With AC as radius and A and B as centers, draw arcs CD and CE. With AB as radius and C as center cut these arcs at E and D. Then $ABCE$ and $ABDC$ are parallelograms. With DB as radius and centers D and E, draw arcs intersecting at G. With CG as radius and D as center draw an arc intersecting the circle C at F. F is the required point on the circumference of C.

Proof:

$$CG^2 = DF^2 = DB^2 - DC^2 = BH^2 + (DC+CH)^2 - DC^2$$
$$= BH^2 + (DC/2)^2 + DC^2 = FC^2 + DC^2.$$

Thus F is the center of the arc AB.

Also solved by *D. L. MacKay* who notes that if the center of the circle is not given, it may be located by the methods of Problem 271, this Magazine, Vol. 13, pp. 392-393; and by *C. L. Steininger* who supplies the reference: *Geometrie du Compas* by Mascheroni (Carette's translation).

No. 330. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries, Kingsville, Texas.

A certain triangle has one side three times another side and the included angle is 60°. Find the triangle with integral sides, the largest of which is less than 150, which is the best possible approximation to the given triangle.

Solution by the *Proposer*.

Let the sides of the triangle be a, $b=3a$ and c. The law of cosines gives $c^2=a^2+b^2-2ab\cos C=a^2+9a^2-3a^2=7a^2$. Thus c/a is approximately $\sqrt{7}$. 2/1, 3/1, 5/2, 8/3, 37/14, 45/17, 82/31, 127/48, $\cdots$, are successive convergents in the continued fraction development of $\sqrt{7}$. We have the desired triangle if we take $a=48$ and $c=127$. Then $C=60°00'17''$.

Editor's Note. Several other solutions were submitted giving close approximations, but not the best approximation. Since $127^2-7\cdot 48^2=1$,

and $7 \cdot 49^2$ differs from the nearest square by 93, it is easy to see that 127/48 gives the best approximation to $\sqrt{7}$ that is possible for $a<50$.

No. 331. Proposed by *Daniel Arany*, Budapest, Hungary.

A tangent is drawn to the circumcircle of A, B, C at the point T. Perpendiculars dropped from A, B, C to this tangent meet it in P_1, P_2, P_3 and the circle in R_1, R_2, R_3.

1. The perpendiculars from R_1, R_2, R_3 upon the sides BC, CA, AB are concurrent at M, a point on the circumcircle.

2. The perpendiculars from P_1, P_2, P_3 upon the sides BC, CA, AB are concurrent at a point S (the orthopole of the tangent at T) on the Simson line of the point M (referred to the triangle ABC).

3. The Simson line of the point T passes through S.

Solution by *Frank Ayres*, Jr., Dickinson College.

Let the coordinates of A, B, $C \equiv A_i$, $(i=1,2,3)$ be denoted by the turns t_i satisfying the equation $t^3-\sigma_1 t^2+\sigma_2 t-\sigma_3=0$, and let the coordinate of T be denoted by the turn T. Then, the perpendicular from A_i meets the tangent at T to the circumcircle in the point P_i, of coordinate $(t_i^2+2t_iT-T^2)/2t_i$, and the circumcircle again in the point R_i, of coordinate $-T^2/t_i$.

The perpendicular from R_i to A_jA_k has the equation

$$z-t_jt_k\bar{z}=\frac{\sigma_3}{T^2}-\frac{T^2}{t_i};$$

the three perpendiculars (obtained by giving i the values 1,2,3) meet in the point M of coordinate σ_3/T^2. This point is on the circumcircle since it satisfies the relation $z\bar{z}=1$. Similarly, the perpendiculars from the points P_i meet in S of coordinate

$$T+\frac{\sigma_1}{2}+\frac{\sigma_3}{2T^2}.$$

The Simson line of M as to ABC has the equation

$$2\sigma_3 z-2T\sigma_3^2\bar{z}=\frac{\sigma_3^2}{T^2}-T^4+\sigma_1\sigma_3-\sigma_2;$$

and that of T the equation $2Tz-2\sigma_3\bar{z}=T^2+\sigma_1T-\sigma_2-\sigma_3/T$. It may be shown readily that S is on both of these lines.

Also solved by *D. L. MacKay*, and the *Proposer*.

No. 335. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Find the locus of the point common to the polar planes, with respect to three given spheres, of a variable point describing a plane perpendicular to the plane passing through the centers of the given spheres.

Solution by *C. W. Trigg*, Los Angeles City College.

Let the centers of the three spheres with radii r, s and t lie in the $XY-$ plane at $(0, 0, 0)$, $(a, b, 0)$ and $(c, d, 0)$, respectively. Let the plane described by the variable point, (f, α, β), be $x=f$. Then the equations of the spheres are

$$x^2+y^2+z^2=r^2,$$
$$(x-a)^2+(y-b)^2+z^2=s^2, \text{ and}$$
$$(x-c)^2+(y-d)^2+z^2=t^2.$$

The equations of the polar planes are, respectively,

$$fx+\alpha y+\beta z=r^2$$
$$(f-a)x+(\alpha-b)y+\beta z=s^2-a^2-b^2+af+\alpha b$$
$$(f-c)x+(\alpha-d)y+\beta z=t^2-c^2-d^2+cf+\alpha d.$$

Now when the parameters α and β are eliminated we secure

$$(ad-bc)(x+f)=d(r^2-s^2+a^2+b^2)-b(r^2-t^2+c^2+d^2).$$

So the locus of the point common to the polar planes is a plane parallel to the given plane.

Also solved by the *Proposer*.

No. 336. Proposed by *V. Thébault*, Le Mans, France.

Find three digits, a, b, c, such that each of the numbers $a00b00c$ and $a00b0c$ is a perfect square.

Solution by *John Ellis Evans*, Rio Grande College.

The square root of $a00b00c$ must be* $\sqrt{a}00\sqrt{c}$. Squaring the latter we have $b=\sqrt{a\cdot c}$. The values of a and c must be selected from 1, 4, 9.

*While this statement is true, it is not entirely self-evdient. Put $a00b00c=N^2$. For $c=5$ or 6, the penultimate digit cannot be 0. Hence $c=0$, 1, 4, or 9. Thus $N^2\equiv c$ mod 1000 implies $N\equiv 0, \pm 1, \pm 2,$ or $\pm 3 \bmod 5^3$. Also we must have

$$1000\sqrt{a}<N<100\sqrt{a01}.$$

There are twenty values of N which satisfy the inequality. All of these, except the values given above, are eliminated by the congruence.—ED.

Thus (a, b, c) must be included among (9, 6, 1) ,(4, 8, 4), (4, 4, 1), (1, 6, 9), (1, 4, 4), (1, 2, 1). Only the first set will make $a00b0c$ also a square. Thus $a=9$, $b=6$, $c=1$ is the unique solution to the problem.

Also solved by *Albert Farnell*, *C. W. Trigg*, and the *Proposer*.

No. 337. Proposed by *W. V. Parker*, Louisiana State University.

Given the triangle $A_1A_2A_3$. Perpendiculars to A_1A_2 at A_1 and A_2A_3 at A_2 meet in P. Perpendiculars to A_1A_2 at A_2 and A_2A_3 at A_3 meet in Q. Prove that the line PQ passes through the circumcenter of $A_1A_2A_3$.

Solution by *John Ellis Evans*, Rio Grande College.

Let a_1 and a_2 be the perpendicular bisectors of A_1A_2 and A_2A_3 respectively.

a_1 intersects PQ at its midpoint since parallel lines intercept proportional segments on two transversals. Likewise, a_2 intersects PQ at its midpoint. The intersection of a_1 and a_2 is the circumcenter of triangle $A_1A_2A_3$ and is the midpoint of PQ. Hence PQ passes through the circumcenter of triangle $A_1A_2A_3$.

Also solved by *Walter B. Clarke*, *Jorge Quijano*, *D. L. MacKay*, *Janet Rung*, *Paul D. Thomas*, *C. W. Trigg*, and the *Proposer*.

PROPOSALS

No. 351. Proposed by *M. S. Robertson*, Rutgers University.

If m is a non-negative integer, find the sum function for

$$\sum_{n=1}^{\infty} \frac{(n+z)^m}{(n+1)!} .$$

No. 352. Proposed by *Howard D. Grossman*, New York, N. Y.

If each side of a triangle be divded into n equal parts and through the points of division lines be drawn inside the triangle parallel to each side, then the total number of triangles in the whole figure is the integer nearest to $n(n+2)(2n+1)/8$.

No. 353. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Given any homogeneous polynomial $f(x,y)$ of degree n with real coefficients which satisfies the Laplace equation $f''_x+f''_y=0$; show that $f(x,y)=0$ represents n lines which make angles of π/n with one another.

No. 354. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a scalene triangle having a median, an altitude, and an external angle bisector concurrent.

No. 355. Proposed by *V. Thébault*, Le Mans, France.

Show that the three-digit number 111 is not a perfect square in any system of numeration. Is the same true of the five-digit number 11111 ?

No. 356. Proposed by *C. W. Trigg*, Los Angeles City College.

1. If a line be divided into n equal segments, the sum of the squares of the lines joining any point, P, to the extremities of the segments is equal to $(n+1)/2$ times the sum of the squares of the extreme joins diminished by $n(n^2-1)/6$ times the square of one of the segments.

2. From (1) show that the sum of thes quares of the rays joining the vertex of the right angle to the points of n-section of the hypotenuse of a right triangle is equal to $(n-1)(2n-1)/6n$ times the square of the hypotenuse.

Bibliography and Reviews

Edited by
H. A. Simmons

Development of the Minkowski Geometry of Numbers. By Harris Hancock. The Macmillan Co., New York, 1939. XIV+839 pages; $12.00.

There are parts of number theory essentially geometric in character which become fully clear and understandable only when viewed from a geometric standpoint. That the reduction theory of quadratic forms is nothing but geometry of parallelepipedal or lattice systems of points in spaces of several dimensions was already clearly understood by Gauss and Dirichlet. Minkowski was, however, the first man who created a large body of doctrines, beautiful in its simplicity and rich in arithmetical applications which he himself called "geometry of numbers". About fifty years ago Minkowski began his memorable investigations in the geometry of numbers. Besides several separate papers, each of which can be regarded as a masterpiece, Minowski published two books devoted to the exposition of his investigations: "Geometrie der Zahlen" 1896, and "Diophantische Approximationen", 1907. While the second book presents a very elegant intuitive approach to the subject, the first is written in an extremely abstract and abstruse manner, which makes it rather difficult reading, Besides, it represents only a part of the work as originally planned by Minkowski and never completed in his lifetime for reasons best known to himself.

Professor Hancock in his new voluminous publication, intends, as he states in the introduction, to present a detailed exposition of the entire work of Minkowski in the geometry of numbers, limiting the exposition, with a few exceptions, strictly to what was done by Minkowski. The book is divided into twenty chapters. The first chapter presents in an intuitive manner some of Minkowski's main theorems on convex bodies in two and three-dimensional spaces with arithmetical applications. Chapters II to VI and XIII to XVI contain essentially the material of the "Geometrieder Zahlen". Other chapters, whose titles we shall not reproduce here explicitly, contain the exposition of papers which Minkowski published separately.

On the whole the book appears to us to be an extended commentary on Minkowski's work in the geometry of numbers with elucidations of numerous difficult passages in Minkowski's own writings and with detailed developments of some of the latter's results, published either with slight indications of the proofs or without any proofs at all. Chapter X in this respect is very valuable, for it gives in extenso the detailed proof by Dr. Pepper of the generalized continued fractions algorithm published by Minkowski without proof in 1896.

Evidently it was the intention of the author to follow Minkowski very closely and to avoid using the numerous contributions to geometry of numbers by later investigators even though in some instances they would have simplified the exposition considerably. One misses, for instance, a very general and useful theorem of H. F. Blichfeldt from which Minkowski's main theorem on convex bodies and many other results can easily be derived. Chapters XIII to XVI reproduce that part of "Geometrie der Zahlen" which is most difficult to read, although Minkowski's method is undoubtedly very ingenious. The results arrived at so painfully in these chapters can, however be established on a few pages as Davenport has recently shown.

Although the blended exposition of Minkowski's own work with more recent contributions to geometry of numbers seems desirable, yet one should not forget the immensity of such a task. The book, as it is, is a valuable contribution to mathematical literature, and we may express the hope that it will attract the attention of the younger generation of mathematicians to the vast and beatiuful field of geometric number theory opened by Minkowski.

Stanford University J. V. USPENSKY

Mathematical Methods in Engineering. By Theodore V. Karman and Maurice A. Biot. McGraw-Hill Book Co., New York, 1940. xii+505 pages.

In this book an attempt is made to develop the mathematical approach to the solution of a set of representative engineering problems. The mathematical topics especially stressed are: linear differential equations (with a whole chapter devoted to Bessel functions), solution of algebraic equations (iteration processes, Newton's method, root-squaring), Fourier series and integrals, operational calculus (Carson's integral equation, Bromwich's integral) and finite differences.

Four chapters are given to mechanics (Lagrange's equations, small oscillations of conservative and non-conservative systems, calculation of normal modes by use of matrices). In two chapters differential equations and Fourier series are applied to the theory of structures (deflections and vibrations of beams, buckling of columns). Mechanical and electrical networks are studied by use of complex numbers, Fourier series, and the operational calculus. A final chapter indicates applications of the calculus of finite differences to such varied problems as the theory of continuous beams., buckling of a truss, voltage drop in a chain of electric insulators, critical speeds of a multicylinder engine, waves in a mechanical chain, and wave filters.

At the end of each chapter references are given for further reading. After Chapter XI there appears a list of sixty four definitions of mathematical terms.

Objections could be raised concerning several of the statements about the theory of definite integrals as presented in Chapter I. On pages 13, 21, 22, 114, 116 the usual convention of interpreting the square root sysmbol to mean the principal value is not followed. On page 32 the word *imaginary* is used instead of *complex*. In at least seven integrals the lower limit has been omitted. Some confusion results from the use of a right-handed system of reference axes and the clockwise rotation of pages 73, 74 (vector product). In the description of Newton's method the role played by $f'(x)$, $f''(x)$ is not mentioned.

This book is worthy of the attention of those interested in applied mathematics. It presents a definite challenge to the engineering student; can he apply his mathematical knowledge to new problems in his own field, or is he confined to the limits of his handbook?

Virginia Military Institute W. E. BYRNE

Advanced Calculus. By I. S. Sokolnikoff. McGraw-Hill, New York 1939. x+446 pages.

Arranging and conducting a satisfactory course in advanced calculus seems to be as difficult as it is interesting. This is especially true of the three-hour course given in one semester. The large group of topics which seem to belong naturally to such a course would in itself present serious difficulties; but in addition to this, one hopes to give the student a proper introduction to the concise reasoning used in analysis. To anyone

who has considered writing a book on the subject it is evident that it is hardly possible to attain all the objectives satisfactorily. But it does seem to be essential to have a text which properly displays sound reasoning in all the topics presented.

Sokolnikoff's book is outstanding in that respect. The developments are accurate and clear and, whenever it is practical to make them so, they are complete. Otherwise it is expressly stated as to just what theorems are needed to complete the arguments and where their proofs can be found. As a result any difficulties encountered by the student in following the argument are usually remedied by more careful reading.

The author is quite successful in his presentation of the precise definitions of limit and continuity, in the first chapter. He could have added, to advantage, a few explicit statements about the continuity of the elementary functions.

The topics covered in the chapters following are: derivatives and differentials, functions of several variables including composite and implicit functions, definite integrals, line integrals, multiple integrals, infinite series, power series and their applications and Fourier series. The three chapters on infinite series covering over a hundred pages carry the subject considerably farther than is customary in advanced calculus, with gratifying results. Again the short chapter on Fourier series is unusually well presented.

The absence of any treatment of differential equations seems not only justified but desirable in such a book. The omission of short chapters on complex variables, vector operations, and calculus of variations, however, may be objectionable to some teachers of full-year courses.

The book shows signs of haste in the proof reading and the arrangement of problems. Typographical errors are rather numerous, and there are some duplications in the problems. It seems that the problems should not only be made more plentiful, and more illustrative and interesting, but they should be more carefully graded. Answers have been given to only a few; so the reviewer has found it necessary to furnish his students with answers to many of the problems.

University of Michigan. R. V. CHURCHILL.

Complex Variable and Operational Calculus. By N. M. McLachlan. Cambridge University Press, 1939. xi+355 pages. $6.50.

This volume was written for applied mathematicians in an attempt to remove some of the mystery with which the Heaviside operational calculus has been associated in the past. Part I, which deals with complex variables (the theory of residues and the Bromwich contour being heavily stressed), serves as an introduction. Part II takes up the Laplace transform and its inversion by the Bromwich-Wagner integral; the theory is applied to the solution of ordinary differential equations with arbitrary initial conditions. Part III treats technical applications and partial differential equations. Part IV gives some of the proofs and discussions omitted from the previous parts of the book, a list of formulas and references to 222 scientific papers and books, and an index. Each of the first three parts is preceded by a short introductory statement concerning the concepts to be mastered for full appreciation of the power of the methods.

There are a few obvious errors. The figures are clear and numerous. Mathematicians might prefer more rigor; however, the author has made a very successful effort to strike "a happy mean" between the interests of the mathematician and the requirements of the technologist. The reviewer believes that many technical students will gain an incentive to further critical mathematical study by reading this work. Possibly this field may become as familiar to engineers of the future as complex variable representation in alternating current theory is at present.

Virginia Military Institute W. E. BYRNE

Mathematical Adventures. By Fletcher Durell. Bruce Humphries, Inc., New York 1938. ii+143 pages; $2.00.

The writer states in the foreword: "This book consists in the main of reprints of articles on teaching of mathematics and on related topics which the author has written and published in educational journals at various times during the past ten years. The last two chapters dealing with recreations and the fourth dimension, were given as lectures before the Mathematical Club of Temple University and have not been hitherto published.

The author endeavors to show in twelve chapters that each branch of mathematics is not independent but that each may be a help for the others. The term "cooperative mathematics" is used frequently throughout the book to mean the interchange of processes between the branches of mathematics. This enlivens the whole subject and stimulates a greater interest from the standpoint of the student. The author states that in cooperative mathematics each branch, as arithmetic, algebra, geometry, etc., keeps its own individuality and organization, but at the same time transfers from the other branches and welcomes to itself any detail which may be locally useful.

A list of advantages of cooperative mathematics is given with some elaboration of these several advantages, comparing both the old-line and general mathematics.

The importance of the formula is stressed, but warning is given that its too free use without sufficient understanding is not stimulating to the pupils' imagination.

By a careful use of graphs, facts represented may be learned and grasped more readily and in a more accurate form. Cultural values, such as elementary training in system and order and a keener sense of analysis, may be gained by a generous use of graphs.

Word problems in arithmetic are discussed from the standpoint of analyzing the problems into patterns.

Attention is drawn to the striking outcomes of the wide use of various new tests in educational work to help determine the ability and aptitudes of pupils.

The value-logic or semi-formal method of presentation helps to keep the interest of the pupil and is recommended as most desirable for the senior high school. Here also is an aid to vocational guidance; for when pupils are working in the way discussed they are making an unconscious but vital self-revelation of their talents and aptitudes.

The merits of four types of objective tests commonly in use, namely the *Completion, True-False, Multiple Choice,* and *Matching* tests are the subject of a chapter. In closing, the statement is made that it seems we are only at the beginning of grasping values which may be derived from further investigation and experiment with the above types of exercises.

In the last two chapters several more or less simple recreations are illustrated, showing the absence of an exact dividing line between vocation and recreation, and how mathematical recreations help to develop systematic organization in the mind. Also presented is an easy approach to *Fourth Dimension* which may stimulate the pupil to further mathematical investigation.

The book, a wholesome discussion of cooperation between the various branches of mathematics, should be in college libraries for education classes to read and ponder over. Much good advice is offered in its pages regarding the teaching of junior high school and high school mathematics.

The James Millikin University. EARL C. KIEFER

The Concepts of the Calculus. By Carl B. Boyer. Columbia University Press, New York, 1939. vi+346 pages.

In this book the author traces the development of calculus concepts from the time of the early Greeks up to the end of the nineteenth century. An introductory chapter indicates the general trend in the formulation of the basic concepts of the calculus. The next six chapters give a more detailed exposition of the contributions of various schools of thought. According to Chapter II, *Concepts in Antiquity*, the early Greek schools leaned on a geometric rather than an arithmetic approach. Although the germs of differentiation and integration were present, they did not result in a general theory because of the confusion of abstract and concrete, the lack of the theory of irrational numbers, limits and continuity. The medieval period (Chapter III) was characterized by philosophical speculation concerning infinity, infinitesimals, continuity, motion and variation. Chapter IV, *A Century of Anticipation*, deals with the improvements due to literal symbolism, the innovations in method of Stevin, Valerio, Kepler, Torricelli, St. Gregory of Vincent, Tacquet, Wallis, Robertval, Fermat and Descartes. In Chapter V, *Newton and Leibnitz*, an interesting account is given of the confusion concerning infinitesimals, differentials, and limits. The indecision and lack of clarity of the inventors of the calculus caused much doubt in their followers, but did not prevent great success in application. Chapter VI, *The Period of Indecision*, describes some of the controversies about the logical foundations of calculus. Chapter VII, *The Rigorous Formulation*, shows how the presentation that one finds in modern elementary texts came into being. Bolzano and, later, Cauchy gave precise definitions of limit, continuity, and derivative. Cauchy stated rigorously the fundamental theorem of integral calculus. Abel, Bolzano, Cauchy, and Gauss put the theory of infiite series on a satisfactory basis. Weierstrass succeeded in giving a further arithmetization of analysis. Other advances were made by the introduction of the Dedekind cut and the theory of sets. The final chapter, a discussion of the tendencies manifest in the growth of the calculus, is followed by a twenty-five page bibliography.

Both students and instructors alike should find this volume of interest, since many of the conceptual difficulties of early mathematicians still trouble students in classrooms every day. A more detailed account, along the same line, of the growth of calculus from the Cauchy-Riemann-Weierstrass period to the present would prove of v alue to all of us.

Virginia Military Institute. W. E. BYRNE.

The Meaning of Mathematics. Second Edition. By Cassius Jackson Keyser. Scripta Mathematica, New York, 1939. 14 pages. Price 15 cents.

This article is concerned with three things: (1) the nature of Mathematics; (2) the nature of mathematical Applications; (3) the nature of the Bearings of mathematics. (The capitals are the author's.)

An insight into the nature or individuality of mathematics may be gained by contemplating mathematics as a kind of intellectual enterprise. or as a body of knowledge, or as a type of thinking.

To the last of these the author devotes about eight pages. He points out that in a postulational discourse the primitive terms are undefined and are therefore variables. Since the postulates contain these variables, they are not propositions but propositional functions, a term borrowed from Bertrand Russell. Likewise the theorems, being defined by use of the variables, are also propositional functions. By the relation of Logical Implication,—that unique relation which "is the heart of heart of mathe-

matical thinking",—these two sets of propositional functions are bound inseparably together. The resulting assertion is called a hypothetical proposition. The totality of these hypothetical propositions constitutes, in a particular mathematical discourse, a Hypothetical Doctrinal Function which is "the spiritual burden or message of the discourse."

Distinction is made between the hypotheticals of mathematics and the categoricals of science.

Applications of mathematics are always concerned with some particular branch of subject matter, while mathematics itself is not concerned with any kind of subject matter.

The author achieves his purpose in a satisfying way. The article should prove helpful and stimulating to many students of mathematics.

Northern Illinois State Teachers College. NORMA STELFORD.

First Year College Mathematics. By Louis C. Plant and Theodore R. Running. American Book Company, 1939. vi+424 pages.

This text represents another thrust in that persistent and slowly gaining drive to unify the teaching of the subjects in freshman mathematics; that is, to teach algebra, trigonometry, analytical geometry, and the fundamentals of calculus according to a pattern of analysis, rather than as four logically unrelated subjects. Certain texts in this class have accomplished a fairly homogeneous composition of the above subjects. In these the concepts of calculus, and the techniques of analytic geometry are introduced very early, and are cultivated throughout. That the leaven is not thoroughly mixed in the text under consideration is made evident by the analysis of contents which follows.

The main text, 377 pages, is devoted to the following topics: a brief introduction of graphs and functional notation followed by trigonometry, logarithms and principles of computation, six chapters or 84 pages; linear, logarithmic, and exponential functions, followed by determinants of order three with applications to alignment charts, three chapters or 78 pages; approximate integration, approximate differentiation, solutions of equations, differentials and derivatives of simple algebraic functions, four chapters; or 87 pages; the differential calculus of exponential, logarithmic, and trigonometric functions, two chapters or 22 pages; formal integration and applications, one chapter or 14 pages; conic sections, three chapters or 46 pages; curve fitting, permutations and combinations, and probability, three chapters or 28 pages; analytic geometry of three dimensions, four chapters or 21 pages. The appendices, of 40 additional pages, contain a review of elementary algebra, a battery of formulas, and such numerical tables as are needed.

In Chapter 2, on functional notation, the topics *variation*, and *ratio and proportion* are treated separately, with no mention of the fact that the one leads to the other. A list of eight problems on ratio and proportion contains seven problems on plane geometry.

The theory of plane trigonometry is developed largely through lists of exercises. Such matters as the reduction of functions of large angles to functions of acute angles; the developments of addition formulas, multiple angle formulas, and half angle formulas are listed as numbered exercises. While the more important results of these developments are displayed in bold faced type, it would not be easy under such an arrangement for a teacher to make well balanced assignments of theoretical problems and practice problems. It would be difficult to keep the theory intact unless the entire list of problems were assigned.

Aside from a neeed for more displayed results, the chapter on linear functions appears to be well done. The use of the *least squares* method of fitting a straight line is to be commended.

An interesting geometric treatment of infinite geometric series occurs in Chapter 8. A student would need to be convinced that the series so represented is not finite.

The application of third order determinants to the construction of alignment charts is also novel. The average college freshman may be expected to find this chapter difficult.

The reviewer admits a prejudice in favor of the study of rates of change of functions and integration by means of straightforward calculus methods, carefully explained, and extended over an ample interval of time. To him, the approximate methods of integration and differentiation used in Chapters 10 and 11 tend to bring dullness to the subject. The ponderous lists of data require laborious attention to analyze them. In the language of page 177 what is called a "square unit of area" is actually a rectangle with sides in the ratio of 40 to 1.

While developing the calculus of exponential functions in Chapter 14, the authors make a commendable emphasis of the scientific importance of such functions.

In the chapter on formal integration, the determination of the area under a curve by means of a definite integral is explained in a lame fashion. The development of Simpson's rule as an immediate sequel would seem to imply that this rule is of basic importance.

The three chapters devoted to conic sections draw no comment except that Simpson's rule is applied again, first on the area of a segment of a circle, and later on the area of an ellipse.

The chapter on curve fitting is limited to problems which can be reduced to the fitting of straight lines.

The word "convergent" does not appear in the index, and the reviewer finds no definition of it; yet it is used in connection with the exponential series.

Answers to problems are not given.

In summary, the text has an honest claim for a fair amount of newness in methods and materials. There is, however, a certain lack of fluidity in the organization of the text. Approximate methods, probably intended for work on inductive problems, cast a long shadow on the deductive analysis.

Central Y. M. C. A. College of Chicago. L. D. GORE.

A Short History of Science. By W. T. Sedgwick and H. W. Tyler. Revised by H. W. Tyler and R. P. Bigelow. The Macmillan Company, New York, 1939. xxi+512 pages. $3.75.

The particular merit of this *Short History of Science* undoubtedly lies chiefly in bringing together in one volume, largely in chronological sequence, the successive achievements in the varied fields of scientific activity. Simply recording in order of appearance notable scientific contributions to human progress reveals and emphasizes the dependence of the separate developments on the general advance in intellectual activity.

The history of science in five hundred pages is no more possible than the history of all civilization within that compass. For true appreciation of mathematical developments, intensive training in mathematics and many years of study are necessary; the same is true of biology, of physics, of chemistry, of astronomy, and of medicine. No one can reasonably expect that two or three authors, not professional historians of

science, should be able to give a correct synthesis of the historical development in all these fields. The present work does cite the chief names and the major achievements in science. In the notes at the end of each chapter and particularly in the *Short List of Reference Books* (pp. 487-500) many of the modern authoritative works in the history of science are indicated.

Particularly commendable in this work is the number and variety of illustrations of vital material in the development of scientific ideas, rather than portraits of men named. In the new edition there are some 81 illustrations, as opposed to 14 in the first edition. The old instruments and machines give definite and indisputable connection with the past.

In mathematics there are five quotations or references taken from the history of mathematics by W. W. R. Ball, who had a positive genius for incorrect statement, and only one quotation from Florian Cajori, a modern historian; Gow's work of 1884 on Greek mathematics is cited six times, while Sir T. L. Heath is cited three times, and Paul Tannery not at all; Heath and Tannery have been for twenty years the ultimate authorities on Greek mathematics. In Arabic Hindu, and medieval science the superlative importance of Sarton's *Introduction to the History of Science* should have been indicated and possibly some quotations introduced.

The multitude of quotations for a work with so large a theme emphasizes to the serious reader the necessity for further information and indicates some of the sources of such material.

The revision introduces, as indicated, far more illustrations, important modifications of the text to indicate modern advance, and important additions concerning recent works on the history of science in the "References for Reading" at the end of chapters and in the final bibliography.

The new edition will be found to be more widely useful and stimulating than the first edition.

University of Michigan. LOUIS C. KARPINSKI.